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Weighted Graph Colorings

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Abstract We study two weighted graph coloring problems, in which one assigns q colors to the vertices of a graph such that adjacent vertices have different colors, with a vertex weighting w that either disfavors or favors a given color. We exhibit a weighted chromatic polynomial $Ph(G,q,w)$ associated with this problem that generalizes the chromatic polynomial $P(G,q)$. General properties of this polynomial are proved, and illustrative calculations for various families of graphs are presented. We show that the weighted chromatic polynomial is able to distinguish between certain graphs that yield the same chromatic polynomial. We give a general structural formula for $Ph(G,q,w)$ for lattice strip graphs G with periodic longitudinal boundary conditions. The zeros of $Ph(G,q,w)$ in the q and w planes and their accumulation sets in the limit of infinitely many vertices of G are analyzed. Finally, some related weighted graph coloring problems are mentioned.

Keywords Graph coloring · Potts model

1 Introduction

Recently we have formulated two weighted graph coloring problems in which one assigns q colors to the vertices of a graph such that adjacent vertices (i.e., vertices connected by an edge of the graph) have different colors, with a vertex weighting w that either disfavors (for $0 \le w < 1$) or favors (for $w > 1$) a given color [1]. We label these with the abbreviations DFCP and FCP for disfavored-color and favored-color weighted graph vertex coloring problems. Since all of the colors are, *a priori*, equivalent, it does not matter which color one takes to be subject to the weighting. In the present paper we study these interesting

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weighted graph coloring problems in detail. An assignment of q colors to the vertices of a graph G, such that adjacent vertices have different colors, is called a "proper q-coloring" of the vertices of G. We analyze the properties of an associated weighted chromatic polynomial that we denote $Ph(G,q,w)$, which generalizes the chromatic polynomial $P(G,q)$ and constitutes a w-dependent measure, extended from the integers to the real numbers, of the number of proper q-colorings of the vertices of G. In the weighted graph coloring problem, with $q \in \mathbb{N}_+$ being the number of colors, for a given graph G, $P(G,q)$ is a map from $\mathbb{N}_+$ to $\mathbb{N}$, while $Ph(G,q,w)$ is a map from $\mathbb{N}_+ \times I$ to $\mathbb{R}$, where I denotes the DFCP interval $0 \le w < 1$ or the FCP interval $w > 1$. In both cases, one can formally extend the domain of q and w to $\mathbb{R}$ or, indeed, $\mathbb{C}$, and the latter extension is, in fact, necessary when one analyzes the zeros of $P(G,q)$ or $Ph(G,q,w)$, respectively. The polynomial $Ph(G,q,w)$ is equivalent to the partition function of the q-state Potts antiferromagnet on the graph G in an external magnetic field H, in the limit where the spin-spin exchange coupling becomes infinitely strong, so that the only spin configurations contributing to this partition function are those for which spins on adjacent vertices are different [1, 2]. There has been continuing interest in the Potts model and chromatic and Tutte polynomials for many years; reviews of the Potts model include [3–6] and reviews of chromatic and Tutte polynomials include [7–15].

There are a number of motivations for this study. One is the intrinsic mathematical interest of these two new kinds of graph coloring problems. A second stems from the equivalence to the statistical mechanics of the Potts antiferromagnet in a magnetic field. A third is the fact that these weighted graph coloring problems have physical applications.[1]

We remark on some special cases of $Ph(G,q,w)$. The case $w = 1$ is equivalent to the usual (unweighted) chromatic polynomial, $P(G,q)$, counting the number of proper q-colorings of the vertices of G:

$$Ph(G,q,1) = P(G,q). \tag{1.1}$$

The chromatic number of G, denoted $\chi(G)$, is the minimal number of colors for which one can carry out a proper q-coloring of the vertices of G. For $w = 0$, one is complete forbidden from assigning the disfavored color to any of the vertices, so that the problem reduces to that of a proper coloring of the vertices of G with $q-1$ colors without any weighting among these $q-1$ colors, which is thus described by the usual unweighted chromatic polynomial $P(G,q-1)$:

$$Ph(G,q,0) = P(G,q-1). \tag{1.2}$$

Thus, the DFCP, described by $Ph(G,q,w)$ with $0 \le w \le 1$, may be regarded as interpolating between $P(G,q)$ and $P(G,q-1)$. In the FCP, as w increases above 1 to large positive values, the favored weighted of one color is increasingly in conflict with the strict constraint that no two adjacent vertices have the same color. Hence, the FCP involves frustration in the technical sense of statistical mechanics, i.e. mutually conflicting tendencies built into the system.

2 Definitions and Some Basic Properties

2.1 Relation with Potts Model in an External Magnetic Field

Consider a graph $G = (V,E)$, defined by its set of vertices V and edges (= bonds) E. A spanning subgraph $G' \subseteq G$ is defined as the subgraph containing the same set of vertices

[1] For more details, see the online version of this paper [16].

V and a subset of the edges; $G' = (V, E')$ with $E' \subseteq E$. For a graph G we denote the number of vertices, edges, and connected components as $n(G)$, $e(G)$, and $k(G)$, respectively. Where no confusion can result, we shall often abbreviate $n(G)$ as simply n. We further denote the connected subgraphs of a spanning subgraph G' as G'_i, $i = 1, \ldots, k(G')$. To obtain an expression for $Ph(G, q, w)$, we make use of the fact that it is a special case of the partition function for the q-state Potts model in an external magnetic field in the limit of infinitely strong antiferromagnetic spin-spin coupling. In thermal equilibrium at temperature T, the general Potts model partition function is given by

$$Z = \sum_{\{\sigma_n\}} e^{-\beta \mathcal{H}} \tag{2.1}$$

with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j} - H \sum_{\ell} \delta_{\sigma_\ell, 1}, \tag{2.2}$$

where i, j, ℓ label vertices (sites) in G, $\sigma_i = 1, \ldots, q$ are classical spin variables on these vertices, $\beta = (k_B T)^{-1}$, and $\langle ij \rangle$ denote pairs of adjacent vertices. Without loss of generality, we have taken the magnetic field H to single out the spin value $\sigma_i = 1$. Let us introduce the notation

$$K = \beta J, \qquad h = \beta H, \qquad y = e^K, \qquad v = y - 1, \qquad w = e^h. \tag{2.3}$$

Thus, the physical ranges of v are $v \ge 0$ for the Potts ferromagnet, and $-1 \le v \le 0$ for Potts antiferromagnet. The weighted chromatic polynomial is then obtained by choosing the antiferromagnetic sign of the spin-spin coupling, $J < 0$ and taking $K \to -\infty$ while keeping $h = \beta H$ fixed. Since $K = \beta J$, the limit $K \to -\infty$ results if one takes $J \to -\infty$ while holding T and H fixed and finite. Alternatively, the limit $K \to -\infty$ can be obtained by taking the zero-temperature limit $T \to 0$, i.e., $\beta \to \infty$, with J fixed and finite and $H \to 0$ so as to keep $h = \beta H$ fixed and finite. The limit $K \to -\infty$ guarantees that no two adjacent spins have the same value, or, in the coloring context, no two vertices have the same color. One sees that in this statistical mechanics context, it is the external magnetic field that produces the weighting that favors or disfavors a given value for the spins σ_i. Positive H gives a weighting that favors spin configurations in which spins have a particular value, say 1, or equivalently, vertex colorings with this value of the color assignment, while negative H disfavors such configurations. For positive and negative H, the corresponding ranges of w are $w > 1$ and $0 \le w < 1$, respectively.

The partition function Z can be written in a manner that does not make explicit reference to the spins σ_i or the summation over spin configurations, but instead as a sum of terms arising from spanning subgraphs $G' \in G$. The formula, obtained by F.Y. Wu, is [17]

$$Z(G, q, v, w) = \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} \left(q - 1 + w^{n(G'_i)}\right). \tag{2.4}$$

This can be understood by writing (2.1) with (2.2) as

$$Z = \sum_{\{\sigma_n\}} \left[\prod_{\langle ij \rangle} (1 + v\delta_{\sigma_i \sigma_j})\right]\left[\prod_{\ell} e^{h\delta_{\sigma_\ell, 1}}\right]. \tag{2.5}$$

If $h = 0$, then each edge of a particular G' gives a contribution of v and represents a spin configuration in which the spins on the ends of this edge have the same value. The spins in each component of G' are connected by edges, so they all have the same value, and there are q possibilities for this value. In this case, from (2.5) one sees that the resultant term in summand of (2.4) is simply $v^{e(G')}q^{k(G')}$. If $h \neq 0$, all of the spins in each connected component G'_i of G' have either the value $\sigma_i = 1$ or all of these spins have one of the other $q-1$ values. If they all have the value 1, then each vertex in this G'_i gives a contribution of w, so from G'_i one gets the contribution $w^{n(G'_i)}$, while if they all have one of the other $q-1$ values, the contribution is 1. In general, therefore, the contribution of the component G'_i in G' is $(q-1+w^{n(G'_i)})$. Taking account of all of the $k(G')$ components in each G' gives the factor $\prod_{i=1}^{k(G')}(q-1+w^{n(G'_i)})$, which is then summed over all $G' \subseteq G$. The Wu formula (2.4) is a generalization of the Fortuin-Kasteleyn formula for the zero-field model [18]. The original definition of the Potts model, (2.1) and (2.2), requires q to be in the set of positive integers $\mathbb{N}_+$. This restriction is removed by (2.4). Equation (2.4) shows that Z is a polynomial in the variables q, v, and w, hence our notation $Z(G,q,v,w)$.

In the special case of zero external magnetic field, $H = 0$, whence $w = 1$, one has the reduction to the Fortuin-Kasteleyn cluster formula [18]

$$Z(G,q,v,1) = \sum_{G' \subseteq G} v^{e(G')}q^{k(G')}. \tag{2.6}$$

This zero-field Potts model partition function is equivalent to the Tutte polynomial $T(G,x,y)$, defined by

$$T(G,x,y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)}(y-1)^{c(G')}, \tag{2.7}$$

where $c(G')$ is the number of linearly independent cycles in G', satisfying $c(G') = e(G') + k(G') - n(G')$, and

$$x = 1 + \frac{q}{v}. \tag{2.8}$$

(We remark that $k(G') - k(G')$ and $c(G')$ are the rank and co-rank of G'.) The equivalence is given by the relation

$$Z(G,q,v) = (x-1)^{k(G)}(y-1)^{n(G)}T(G,x,y). \tag{2.9}$$

In [1] we defined a generalization of the Tutte polynomial,

$$U(G,x,y,w) = (x-1)^{-k(G)}(y-1)^{-n(G)} \sum_{G' \subseteq G} (y-1)^{e(G')} \times \prod_{i=1}^{k(G')}(xy - x - y + w^{n(G'_i)}). \tag{2.10}$$

This function satisfies $U(G,x,y,w) = (x-1)^{-k(G)}(y-1)^{-n(G)}Z(G,q,v,w)$ and reduces to the Tutte polynomial if $w = 1$: $U(G,x,y,1) = T(G,x,y)$.

The $K \to -\infty$ limit that yields the weighted chromatic polynomial is equivalent to $v = -1$, so

$$Ph(G,q,w) = Z(G,q,-1,w). \tag{2.11}$$

Hence, a constructive formula for $Ph(G,q,w)$ is

$$Ph(G,q,w)=\sum_{G'\subseteq G}(-1)^{e(G')}\prod_{i=1}^{k(G')}\big(q-1+w^{n(G'_i)}\big). \tag{2.12}$$

For the special case $h=0$, i.e., $w=1$, one thus has the result of (1.1). The limit $h\to-\infty$, i.e., $w\to 0$, effectively removes one of the possible values of the dynamical variables σ_i, or equivalently, in the spanning subgraph formula (2.4), one of the values of q, so

$$Z(G,q,v,0)=Z(G,q-1,v,1). \tag{2.13}$$

The special case of this for $v=-1$ is (1.2). It follows that each of the zeros of $Ph(G,q,1)\equiv P(G,q)$ in the complex q plane shifts to the right by one unit as one replaces the value $w=1$ by $w=0$. In the limit as $n\to\infty$, the accumulation set of the zeros, $\mathcal{B}_q$ also is replaced by its identical image shifted to the right in the q plane as one replaces $w=1$ by $w=0$.

2.2 Results for Graphs with Loops, Multiple Edges, and Multiple Components

If G has any loop, defined as an edge that connects a vertex to itself, then a proper q-coloring is impossible. This is because such a q-coloring requires that any two adjacent vertices have different colors, but since the vertices connected by an edge are adjacent, the presence of a loop in G means that a vertex is adjacent to itself. Thus,

$$Ph(G,q,w)=0 \quad \text{if } G \text{ contains a loop.} \tag{2.14}$$

Hence, to avoid having $Ph(G,q,v)$ vanish trivially, we shall restrict our analysis in this paper to loopless graphs G. Accordingly, in the text below, where $G=(V,E)$ is characterized as having a non-empty edge set $E\neq\emptyset$, it is understood that E does not contain any loops.

Another basic property of a chromatic polynomial is that as long as two vertices are joined by an edge, adding more edges connecting these vertices does not change the chromatic polynomial. This is clear from the fact that the chromatic polynomial counts the number of proper q-colorings of the vertices of G, and the relevant condition—that two adjacent vertices must have different colors—is the same regardless of whether one or more than one edges join these vertices. Let us define an operation of "reduction of multiple edge(s)" in G, denoted $R_E(G)$, as follows: if two vertices are joined by a multiple edge, then delete all but one of these edges, and carry out this reduction on all edges, so that the resultant graph $R_E(G)$ has only single edges. Then if G is a graph that contains one or more multiple edges joining some set(s) of vertices,

$$P(G,q)=P(R_E(G),q). \tag{2.15}$$

Since the same proper q-coloring condition holds for the weighted chromatic polynomial, we have

$$Ph(G,q,w)=Ph(R_E(G),q,w). \tag{2.16}$$

Moreover, if G consists of two disjoint parts, G_1 and G_2, then $Ph(G,q,w)$ is simply the product $Ph(G,q,w)=Ph(G_1,q,w)Ph(G_2,q,w)$. Hence, without loss of generality, we will generally restrict to connected graphs G.

2.3 General Structural Properties of $Ph(G, q, w)$

Here we prove some general structural properties of $Ph(G, q, w)$ that hold for an arbitrary graph G. As discussed above, to avoid having $Ph(G, q, w)$ vanish trivially, we take G to be loopless, and without loss of generality, we assume that G is connected. We first apply the proper q-coloring condition to analyze properties of $Ph(G, q, w)$ for $q \in \mathbb{N}_+$. Since this proper q-coloring condition cannot be met for integer $q = 1, \dots, \chi(G) - 1$, it follows that

$$Ph(G, q, w) \text{ contains a factor } \prod_{j=1}^{\chi(G)-1} (q - j). \tag{2.17}$$

Provided that $G = (V, E)$ has at least one edge, i.e., $E \neq \emptyset$, the proper q-coloring condition cannot be satisfied if $q = 1$. Hence, a corollary of (2.17) is

$$\text{If } E \neq \emptyset, \text{ then } Ph(G, q, w) \text{ contains a factor } (q - 1). \tag{2.18}$$

We can show that if $w = 0$, then the factor $(q - 1)$ is present even if G does not contain any edge. Using our previous result that $Z(G, q, v, 0) = Z(G, q - 1, v, 1)$ and the fact that $Z(G, q, v, 1)$ has a factor of q, we obtain the result that $Z(G, q, v, 0)$ contains the factor $q - 1$ and hence $Ph(G, q, 0)$ contains a factor $(q - 1)$. More generally, since $Ph(G, q, 0) = P(G, q - 1)$ and $P(G, q - 1)$ vanishes for integer $q = 1, \dots, \chi(G)$, it follows that

$$Ph(G, q, 0) \text{ contains a factor } \prod_{j=1}^{\chi(G)} (q - j). \tag{2.19}$$

Substituting $q = 0$ in (2.4) and using the factorization

$$w^{n(G'_i)} - 1 = (w - 1) \sum_{\ell=0}^{n(G'_i)-1} w^\ell \tag{2.20}$$

proves that

$$Z(G, 0, v, w) \text{ contains a factor of } (w - 1). \tag{2.21}$$

Setting $v = -1$, we thus deduce that [1]

$$Ph(G, 0, w) \text{ contains a factor of } (w - 1). \tag{2.22}$$

It is convenient to define the notation

$$\tilde{q} = q - 1, \qquad \tilde{w} = w - 1. \tag{2.23}$$

From (2.4), it follows that we can write $Z(G, q, v, w)$ in several equivalent ways:

$$\begin{aligned} Z(G, q, v, w) &= \sum_{r,t=0}^{n(G)} \sum_{s=0}^{e(G)} a_{r,s,t} q^r v^s w^t = \sum_{r,t=0}^{n(G)} \sum_{s=0}^{e(G)} b_{r,s,t} q^r y^s w^t \\ &= \sum_{r,t=0}^{n(G)} \sum_{s=0}^{e(G)} c_{r,s,t} \tilde{q}^r v^s w^t = \sum_{r,t=0}^{n(G)} \sum_{s=0}^{e(G)} d_{r,s,t} q^r v^s \tilde{w}^t, \end{aligned} \tag{2.24}$$

where $a_{r,s,t}$, $b_{r,s,t}$, $c_{r,s,t}$, and $d_{r,s,t}$ are integers. Some $a_{r,s,t}$ and $b_{r,s,t}$ can be negative, but, as we showed in [1], the nonzero $c_{r,s,t}$ and $d_{r,s,t}$ are positive. From these equations, one infers corresponding ones for $Ph(G,q,w)$ by setting $v=-1$, i.e., $y=0$. Note that in the polynomial $Z(G,q,v,w)=\sum_{r,t=0}^{n(G)}\sum_{s=0}^{e(G)} b_{rst}q^r y^s w^t$, clearly only the terms with $s=0$ contribute to $Ph(G,q,w)$.

From (2.4), it is evident that the term in $Ph(G,q,w)$ of maximal degree in q, or equivalently, in $\tilde{q}$, arises from the contribution of the spanning subgraph G' with no edges, for which $k(G')=n(G)$. This term is (with $n\equiv n(G)$)

$$(\tilde{q}+w)^n. \tag{2.25}$$

It follows that

$$a_{n,0,0}=b_{n,0,0}=c_{n,0,0}=d_{n,0,0}=1. \tag{2.26}$$

There are $e(G)$ spanning subgraphs G' with one edge, since there are $e(G)$ ways of choosing this edge. Hence (with our restriction to loopless G), the contribution of these G' in (2.4) is

$$e(G)v(\tilde{q}+w^2)(\tilde{q}+w)^{n-2}. \tag{2.27}$$

Expanding the terms in (2.25) and (2.27) in powers of $\tilde{q}$ and w, we find that the term in $Z(G,q,v,w)$ of degree $n-1$ in $\tilde{q}$ is

$$\big(e(G)v+nw\big)\tilde{q}^{n-1}. \tag{2.28}$$

Similarly, expanding the terms in (2.25) and (2.27) in powers of q and w, we find that the term in $Z(G,q,v,w)$ of degree $n-1$ in q is

$$\big(e(G)v+n(w-1)\big)q^{n-1}. \tag{2.29}$$

For our analysis below and for comparisons with chromatic polynomials, it will be convenient to write $Ph(G,q,w)$ as a polynomial in q with w-dependent coefficients, which we denote $\alpha_{G,\ell}(w)$:

$$Ph(G,q,w)=\sum_{j=0}^{n}\alpha_{G,n-j}(w)q^{n-j}. \tag{2.30}$$

From our discussion above, we have

$$\alpha_{G,n}=1, \tag{2.31}$$

and, using also (2.16),

$$\alpha_{G,n-1}=-\big(e(R_E(G))+n(1-w)\big). \tag{2.32}$$

Moreover, from (2.22), it follows that the q^0 term in $Ph(G,q,w)$ contains a factor of $(w-1)$, i.e.,

$$\alpha_{G,0} \text{ contains a factor of } (w-1). \tag{2.33}$$

It is also useful to express $Ph(G,q,w)$ as a polynomial in w with q-dependent coefficients, which we denote $\beta_{G,\ell}(q)$ (there should not be confusion with $\beta=1/(k_BT)$):

$$Ph(G,q,w)=\sum_{j=0}^{d_w(G)}\beta_{G,j}(q)w^j, \tag{2.34}$$

where $d_w(G) \equiv \deg_w(Ph(G,q,w))$ is the (maximal) degree of $Ph(G,q,w)$ in w. This degree, $d_w(G)$ is a G-dependent number less than n. To understand this, we recall that the maximum degree of $Z(G,q,v,w)$ in w is n. This term is $y^{e(G)}w^n = (v+1)^{e(G)}w^n$ and corresponds to all of the vertices having the same color, 1. However, the possibility that all of the vertices have the same color, and, indeed, the possibility that any adjacent vertices have the same color, are excluded for $Ph(G,q,w)$, as is evident from the fact that the coefficient of w^n vanishes for $v=-1$. Hence, $d_w(G) < n$. We shall give this degree below for various families of graphs. Since all of the nontrivial graphs $G=(V,E)$ that we shall consider have at least one edge, i.e., $E \neq \emptyset$, (2.18) shows that for these, $Ph(G,q,w)$ has the factor $(q-1)$. In analyzing zeros of $Ph(G,q,w)$ it will be convenient to separate this factor out, and we thus define, for graphs containing at least one edge,

$$\beta_{G,j}(q) = (q-1)\bar{\beta}_{G,j}(q), \tag{2.35}$$

so that

$$\text{If } E \neq \emptyset, \quad \text{then } Ph(G,q,w) = (q-1)\sum_{j=0}^{d_w(G)} \bar{\beta}_{G,j}(q) w^j, \tag{2.36}$$

where $\bar{\beta}_{G,j}(q)$ are polynomials in q. From (2.34) and (1.2), we obtain the relation

$$Ph(G,q,0) = \beta_{G,0}(q) = P(G,q-1). \tag{2.37}$$

Combining this with (2.19), we have the result that

$$\beta_{G,0} \text{ contains a factor } \prod_{j=1}^{\chi(G)} (q-j). \tag{2.38}$$

Now, $Ph(G,q,1) = \sum_{j=0}^{d_w(G)} \beta_{G,j}$, but also $Ph(G,q,1) = P(G,q)$, so, using the fact that $P(G,q) = 0$ for integer $q = 0, \dots, \chi(G)-1$, we derive the following factorization property for the sum of the $\beta_{G,j}$ coefficients:

$$\sum_{j=0}^{d_w(G)} \beta_{G,j} \quad \text{contains a factor} \quad \prod_{j=0}^{\chi(G)-1} (q-j). \tag{2.39}$$

2.4 Absence of Deletion-Contraction Relation

For a graph G, we denote the graph obtained by deleting an edge $e \in E$ as $G-e$ and the graph obtained by identifying the two vertices connected by this edge e as G/e. The chromatic polynomial satisfies the deletion-contraction relation $P(G,q,v) = P(G-e,q) - P(G/e,q)$. In contrast, for w not equal to 1 or 0, the polynomial $Ph(G,q,w)$ does not, in general, satisfy this deletion-contraction relation. It is of interest to examine the quantities that measure the deviation from such a relation, namely

$$\Delta Ph(G,e,q,w) = Ph(G,q,w) - \big[Ph(G-e,q,w) - Ph(G/e,q,w)\big]. \tag{2.40}$$

We know that $\Delta Ph(G,e,q,w)$ contains a factor $w(w-1)$ since for $w=1$ and $w=0$, $Ph(G,q,w)$ is equal, respectively, to $P(G,q)$ and $P(G,q-1)$, both of which do satisfy the deletion-contraction relation. Furthermore, because of (2.18), if G, $G-e$, and G/e contain at least one edge, then $\Delta Ph(G,e,q,w)$ contains the factor $(q-1)$.

As an illustration, using our explicit calculations given below for n-vertex line graphs L_n and circuit graphs C_n, we find the following results. For the first two graphs, L_3 and C_3, the deletion and contraction on any edge gives the same result, so we need not specify which edge is involved. We find, for any edge e,

$$\Delta Ph(L_3,e,q,w) = \Delta Ph(C_3,e,q,w) = -w(w-1)(q-1) \tag{2.41}$$

and

$$\Delta Ph(C_4,e,q,w) = -w(w-1)(q-1)(q-2). \tag{2.42}$$

For L_4, denoting e_{outer} as either of the two outer edges and e_{mid} as the middle edge, we find

$$\Delta Ph(L_4,e_{mid},q,w) = -w(w-1)(q-1)^2 \tag{2.43}$$

and

$$\Delta Ph(L_4,e_{outer},q,w) = -w(w-1)(q-1)(q+w-2). \tag{2.44}$$

It is straightforward to calculate similar differences $\Delta Ph(G,e,q,w)$ for graphs with more vertices and edges, but these are sufficient to illustrate the absence of a usual deletion-contraction relation for the weighted chromatic polynomial.

2.5 T, P, U, and Ph Equivalence Classes

An important property of the weighted chromatic polynomial $Ph(G,q,w)$ is the fact that it can distinguish between certain graphs that yield the same chromatic polynomial $P(G,q)$. More generally, an important property of the partition function of the Potts model in a nonzero external magnetic field, $Z(G,q,v,w)$, or equivalently, the function $U(G,x,y,w)$ that we defined in [1], is that $Z(G,q,v,w)$ and $U(G,x,y,w)$ distinguish between graphs that yield the same zero-field Potts model partition function, $Z(G,q,v,1)$ or equivalently, Tutte polynomial $T(G,x,y)$. We begin with some definitions. Two graphs G and H are defined as (i) Tutte-equivalent (T-equivalent) if they have the same Tutte polynomial, or equivalently, the same zero-field Potts model partition function, $Z(G,q,v,1)$; (ii) U-equivalent if they have the same $Z(G,q,v,w)$; (iii) chromatically equivalent (P-equivalent) if they have the same chromatic polynomial, $P(G,q)$, and (iv) Ph-equivalent if they have the same weighted chromatic polynomial, $Ph(G,q,w)$.

Let us give some examples. Recall the definition that a tree graph is a connected graph that contains no circuits (cycles). The set of tree graphs $\{G_{tree,n}\}$ with a given number, n, of vertices, forms a Tutte equivalence class, with $T(G_{tree,n},x,y) = x^{n-1}$, or equivalently, $Z(G,q,v,1) = q(q+v)^{n-1}$. However, the Potts partition function in a field, $Z(G,q,v,w)$, or equivalently, the function $U(G,x,y,w)$ is able to distinguish between different tree graphs in a Tutte-equivalence class. For instance, consider the $n=4$ line graph L_4 and star graph S_4 (the graph with one central vertex connected to three outer vertices by corresponding edges). These have the same Tutte polynomial $T(L_4,x,y) = T(S_4,x,y) = x^3$, or equivalently, the same zero-field Potts partition function $Z(L_4,q,v,1) = Z(S_4,q,v,1) = q(q+v)^3$, but the full Potts partition functions, $Z(L_4,q,v,w)$ and $Z(S_4,q,v,w)$ are different (see (3.6) and (3.16) below). Similarly, L_4 and S_4 are chromatically equivalent, with $P(L_4,q) = P(S_4,q) = q(q-1)^3$ as a special case of the result $P(G_{tree,n},q) = q(q-1)^{n-1}$ for any tree graph with n vertices, $G_{tree,n}$. However, from our calculations given below in

(3.15) and (3.19), we find that $Ph(L_4, q, w)$ and $Ph(S_4, q, w)$ are different. We reach the same conclusion for all of the tree graphs that we have studied, i.e., although the set of tree graphs with a given number, n, of vertices, forms a chromatic equivalence class, these graphs have different weighted chromatic polynomials. We will illustrate this below for $n = 5$ and $n = 6$.

A second set of examples involves graphs with multiple edges. Let us assume that G contains one or more multiple edges joining pair(s) of vertices. Such graphs are not Tutte-equivalent, but, as noted above, are chromatically equivalent. Because the same proper q-coloring condition also holds for weighted chromatic polynomials, these graphs are also in the same Ph-equivalence class, as was stated in (2.16). A simple example is provided by the line and circuit graphs with $n = 2$ vertices, L_2 and C_2, the latter of which has a double edge connecting the two vertices. One has

$$Z(L_2, q, v, w) = (q - 1 + w)^2 + v(q - 1 + w^2) \tag{2.45}$$

and

$$Z(C_2, q, v, w) = (q - 1 + w)^2 + v(v + 2)(q - 1 + w^2), \tag{2.46}$$

so that

$$Z(C_2, q, v, w) - Z(L_2, q, v, w) = v(v + 1)(q - 1 + w^2). \tag{2.47}$$

The fact that the difference in (2.47) vanishes for $v = -1$, i.e., that $Ph(L_2, q, w) = Ph(C_2, q, w)$, is a special case of the general result (2.16).

Because of the above-mentioned result that all n-vertex tree graphs are chromatically equivalent, in conjunction with the property that $Ph(G, q, w)$ is a chromatic polynomial for $w = 1$ and $w = 0$, it follows that the difference between $Ph(G, q, w)$ and $Ph(H, q, w)$ between two chromatically equivalent graphs G and H must vanish if $w = 1$ or $w = 0$. Since these are all polynomials, it thus follows that the difference $Ph(G, q, w) - Ph(H, q, w)$ must have w and $w - 1$ as factors. Furthermore, if $q = 1$, then

$$Z(G, 1, v, w) = y^{e(G)} w^{n(G)}. \tag{2.48}$$

If G has at least one edge, then $Z(G, 1, v, w) = 0$ if $y = 0$, i.e., $v = -1$, Now in order to be chromatically equivalent, a necessary condition is that two graphs G and H must have the same number of vertices, $n(G) = n(H)$, since the degree in q of $P(G, q)$ is $n(G)$. An elementary property of the chromatic polynomial $P(G, q)$, proved by iterative application of the deletion-contraction theorem, is that the coefficient of the $q^{n(G)-1}$ term is $-e(R_E(G))$. Therefore, another necessary condition that two graphs G and H be chromatically equivalent is that $e(R_E(G)) = e(R_E(H))$. Now recall (2.18), according to which if G contains at least one edge, then $Ph(G, 1, w) = 0$. Hence, if G and H are chromatically equivalent, then either (i) neither contains any edges, in which case $Ph(G, q, w) = Ph(H, q, w) = (q - 1 + w)^n$, where $n = n(G) = n(H)$, or (ii) if G, and hence H, contains at least one edge, $Ph(G, 1, w) = Ph(H, 1, w) = 0$. Hence, if G and H are chromatically equivalent and contain at least one edge, then the difference $Ph(G, q, w) - Ph(H, q, w)$ contains the factor $(q - 1)$. These results on the factors of $Ph(G, q, w) - Ph(H, q, w)$ for chromatically equivalent graphs will be evident in our explicit calculations to be presented below.

2.6 On the Weighted Face Coloring Problem for Planar Graphs

Let us consider a planar graph $G = (V, E)$. We recall that the dual of this graph, G^*, is the graph obtained from G by associating a vertex of G^* with each face of G and connecting these vertices of G^* with edges that cross each edge of G. There is thus a 1-1 isomorphism between the vertices, edges, and faces of G and the faces, edges, and vertices of G^*, respectively. A proper q coloring of the faces of G^* is a coloring of these faces with q colors subject to the constraint that no two faces that are adjacent across the same edge have the same color. The (usual, unweighted) chromatic polynomial $P(G,q)$ satisfies a duality property, namely that $P(G,q)$ counts not just the proper q colorings of the vertices of G, but also, and equivalently, the proper q colorings of the faces of G^*. By the same duality property, for a planar graph G, our weighted chromatic polynomial $Ph(G,q,w)$ describes not just the weighted proper q colorings of the vertices of G but also, and equivalently, the weighted proper q colorings of the faces of G^*.

3 Calculations of $Ph(G,q,w)$ for Some Families of Graphs

In this section we give some illustrative explicit calculations of $Ph(G,q,w)$ for various families of graphs. Although we generally consider connected graphs, we note that for the graph N_n consisting of n vertices with no edges,

$$Z(N_n,q,v,w) = Ph(N_n,q,w) = (q-1+w)^n. \tag{3.1}$$

We recall that a tree graph is defined as a connected graph with no circuits. In the following text and in Appendix B we present results for the weighted chromatic polynomials of n-vertex tree graphs with n up to 6.

3.1 Line Graph L_n

The line graph L_n is the graph consisting of n vertices with each vertex connected to the next one by one edge. One may picture this graph as forming a line, whence the name. For $n \ge 2$, the chromatic number is $\chi(L_n) = 2$. In [2] we gave a general formula for $Z(L_n,q,v,w)$, and the special case $v = -1$ determines $Ph(L_n,q,w)$. Let us define

$$T_{Z,1,0} = \begin{pmatrix} q+v-1 & w \\ q-1 & w(v+1) \end{pmatrix} \tag{3.2}$$

$$H_{1,0} = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \tag{3.3}$$

$$u_1 = \begin{pmatrix} q-1 \\ 1 \end{pmatrix} \tag{3.4}$$

and

$$s_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.5}$$

Then

$$Z(L_n,q,v,w) = u_1^T H_{1,0} (T_{Z,1,0})^{n-1} s_1 \tag{3.6}$$

and $Ph(L_n, q, w) = Z(L_n, q, -1, w)$. Since $e(L_n) = n - 1$, we can apply (2.32) to deduce that

$$\alpha_{L_n, n-1} = 1 + n(w - 2). \tag{3.7}$$

From our general formula (3.6), evaluated at $v = -1$ to obtain $Ph(L_n, q, w)$, we can derive some other corollaries concerning coefficients of $Ph(L_n, q, w)$. The maximal degree of $Ph(L_n, q, w)$ in w is

$$\deg_w(Ph(L_n, q, w)) = \left[\frac{n+1}{2}\right], \tag{3.8}$$

where here $[\nu]$ denotes the largest integer less than or equal to $\nu \in \mathbb{R}$. This contrasts with the fact that the highest power of w in $Z(L_n, q, v, w)$ for $v \neq -1$ is n. The reason for this is that spin configurations that would yield terms of degrees less than or equal to n and greater than the maximum in (3.8) are forbidden by the proper q-coloring constraint. If n is odd, say $n = 2m + 1$, the coefficient of the term in $Ph(L_{2m+1}, q, w)$ of highest degree in w, namely the coefficient of the term $w^{(n+1)/2} = w^{m+1}$, is

$$\beta_{L_{2m+1}, m+1} = (q - 1)^m. \tag{3.9}$$

If n is even, say $n = 2m$, the coefficient of the term in $Ph(L_{2m}, q, w)$ of highest degree in w, namely the coefficient of the term $w^{n/2} = w^m$, is

$$\beta_{L_{2m}, m} = (q - 1)^{m-1}\big((m + 1)q - 2m\big). \tag{3.10}$$

The coefficient of the w^0 term in $Ph(L_n, q, w)$ is

$$\beta_{L_n, 0} = (q - 1)(q - 2)^{n-1}. \tag{3.11}$$

We proceed to give some explicit results for $Ph(L_n, q, w)$ for various values of n. The case $L_1 = N_1$ is already covered by (3.1). For the next few cases we list the explicit polynomials below, both in factored form and in the form of (2.30):

$$Ph(L_1, q, w) = q - 1 + w \tag{3.12}$$

$$\begin{aligned} Ph(L_2, q, w) &= (q - 1)\big[q + 2(w - 1)\big] \\ &= q^2 - (3 - 2w)q + 2(1 - w) \end{aligned} \tag{3.13}$$

$$\begin{aligned} Ph(L_3, q, w) &= (q - 1)\big[q^2 + (3w - 4)q + (w - 1)(w - 4)\big] \\ &= q^3 - (5 - 3w)q^2 + (w^2 - 8w + 8)q - (w - 1)(w - 4) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} Ph(L_4, q, w) &= (q - 1)(q + w - 2)\big[q^2 + (3w - 4)q - 4(w - 1)\big] \\ &= q^4 - (7 - 4w)q^3 + 3(w^2 - 6w + 6)q^2 - (7w^2 - 26w + 20)q \\ &\quad + 4(w - 1)(w - 2). \end{aligned} \tag{3.15}$$

Results for tree graphs with $n = 5$ and $n = 6$ vertices are given in Appendix B.

3.2 Star Graphs S_n

A star graph S_n consists of one central vertex with degree $n-1$ connected by edges with $n-1$ outer vertices, each of which has degree 1. For $n \ge 2$, the chromatic number is $\chi(S_n) = 2$. We have derived the following general formula for $Z(S_n, q, v, w)$:

$$Z(S_n, q, v, w) = \sum_{j=0}^{n-1} \binom{n-1}{j} v^j (\tilde{q} + w^{j+1})(\tilde{q} + w)^{n-1-j}, \tag{3.16}$$

where $\tilde{q}$ was given in (2.23). Evaluating this for $v = -1$ yields $Ph(S_n, q, w)$. The term in $Ph(S_n, q, w)$ of maximal degree in w corresponds to a configuration in which all of the outer vertices are assigned the color 1 and the central vertex of the star graph is assigned any of the other $q-1$ colors. For $n \ge 3$ where the star graphs are nondegenerate, this term is thus $(q-1)w^{n-1}$, so that, in particular,

$$\deg_w(Ph(S_n, q, w)) = n - 1. \tag{3.17}$$

(The graph S_2 is degenerate in the sense that it has no central vertex but instead coincides with L_2.) The graph S_3 is nondegenerate, and coincides with L_3. For $n = 2$, the term in $Ph(S_2, q, w)$ of maximal degree in w, namely the coefficient of the term w, is $2(q-1)$. For $n \ge 3$, the coefficient of the term in $Ph(S_n, q, w)$ of maximal degree in w, namely the coefficient of the term w^{n-1}, is $(q-1)$. This is easily understood since it corresponds to the assignment of the color 1 to each of the $n-1$ outer vertices of the star graph S_n, which allows any of the remaining $(q-1)$ colors to be assigned to the central vertex of this graph. Because the number of edges of the star graph is $e(S_n) = n-1$, it follows that

$$\alpha_{S_n, n-1} = 1 + n(w-2). \tag{3.18}$$

This coefficient is equal to $\alpha_{L_n, n-1}$.

As an explicit example, for the graph S_4 we calculate

$$\begin{aligned} Ph(S_4, q, w) &= (q-1)\big[q^3 + 2(2w-3)q^2 + (3w^2 - 14w + 12)q \\ &\quad + (w-1)(w^2 - 5w + 8)\big] \\ &= q^4 - (7-4w)q^3 + 3(w^2 - 6w + 6)q^2 - (-w^3 + 9w^2 - 27w + 20)q \\ &\quad + (1-w)(w^2 - 5w + 8). \end{aligned} \tag{3.19}$$

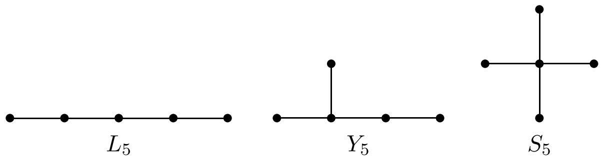
Results for S_n with $n = 5$ and $n = 6$ are given in Appendix B.

3.3 Distinguishing Between Some Chromatically Equivalent Graphs

Using the results given in the text and Appendix B for tree graphs with up to six vertices, we now analyze the differences between the weighted chromatic polynomials for tree graphs that are chromatically equivalent. There are two tree graphs with $n = 4$ vertices, namely, L_4 and S_4. The correspondences with the carbon atom backbones of alkanes of these and other tree graphs considered here are given in the online version of this paper [16]. From (3.15) and (3.19) we find

$$Ph(S_4, q, w) - Ph(L_4, q, w) = (q-1)w(w-1)^2. \tag{3.20}$$

Fig. 1 Tree graphs with $n = 5$



There are three tree graphs with $n = 5$ vertices, as shown in Fig. 1, namely (i) the line graph L_5, (ii) a graph that we denote Y_5, which is obtained by starting with the star S_4 and elongating one of the edges by the addition of another vertex and edge, and (iii) the star graph S_5. We order this list in terms of graphs of increasing maximal vertex degree Δ; one has $\Delta = 2, 3, 4$ for L_5, Y_5, and S_5, respectively. From (B.1), (B.3), and (B.2), we calculate

$$Ph(S_5, q, w) - Ph(L_5, q, w) = (q-1)w(w-1)^2(3q + w - 5) \tag{3.21}$$

$$Ph(S_5, q, w) - Ph(Y_5, q, w) = (q-1)w(w-1)^2(2q + w - 3) \tag{3.22}$$

and thus

$$Ph(Y_5, q, w) - Ph(L_5, q, w) = (q-1)(q-2)w(w-1)^2. \tag{3.23}$$

For these graphs, one observes that the differences between chromatically equivalent graphs have a double zero at $w = 1$. We find that this is also true of the differences between weighted chromatic polynomials of tree graphs with $n = 6$ vertices, as discussed in Appendix B.

3.4 Complete Graphs K_n

The complete graph K_n is the graph with n vertices such that each vertex is connected to every other vertex by one edge. The chromatic number is thus $\chi(K_n) = n$. One has $e(K_n) = \binom{n}{2}$. The simplest two cases coincide with previously discussed graphs, namely the single vertex, $K_1 = L_1$, for which we gave $Ph(L_1, q, w)$ in (3.12), and the $n = 2$ case, for which $K_2 = L_2$ and $Ph(L_2, q, w)$ was given in (3.13). For general $n \geq 2$ we obtain the following theorem:

$$Ph(K_n, q, w) = \left[\prod_{j=1}^{n-1}(q-j)\right](q + n(w-1)). \tag{3.24}$$

Proof To prove this, we begin by observing that because of the proper q-coloring condition, $Ph(K_n, q, w)$ vanishes for all of the integer values $q = 1, \dots, n-1$ and hence must contain the factor $\prod_{j=1}^{n-1}(q-j)$. The proper q-coloring condition also means that only one vertex at most can be assigned the color 1; hence the term in $Ph(K_n, q, w)$ of highest degree in the variable w has degree 1. Since the maximal degree of $Ph(G, q, w)$ in the variable q is $n(G)$, it must be of the form

$$\left[\prod_{j=1}^{n-1}(q-j)\right](aq + bw + c). \tag{3.25}$$

From (2.26), it follows that $a = 1$. From (1.1) we have

$$Ph(K_n, q, 1) = P(K_n, q) = \prod_{j=0}^{n-1}(q-j), \tag{3.26}$$

which implies that $b=-c$, so the last factor in (3.25) is $(q+b(w-1))$. From (1.2) we have

$$Ph(K_n,q,0)=P(K_n,q-1)=\prod_{j=1}^{n}(q-j), \tag{3.27}$$

which implies that $b=n$, so that the additional factor is $(q+n(w-1))$. This proves the result (3.24). □

A corollary of the theorem of (3.24) is that

$$\deg_w(Ph(K_n,q,w))=1 \tag{3.28}$$

and, further, for $n\ge 2$, the term in $Ph(K_n,q,w)$ of maximal degree in w has coefficient

$$\beta_{K_n,1}=n\prod_{j=1}^{n-1}(q-j). \tag{3.29}$$

3.5 Wheel Graphs Wh_n

The wheel graph Wh_n is the graph obtained by joining one central vertex to the $n-1$ vertices of the circuit graph C_{n-1}. (This is the "join" of K_1 with C_{n-1}.) The central vertex can be regarded as forming the axle of the wheel, while the $n-1$ vertices of the C_{n-1} and their edges form the outer rim of the wheel. This is well-defined for $n\ge 3$, and in this range the chromatic number is $\chi(Wh_n)=3$ if n is odd and $\chi(Wh_n)=4$ if n is even. The graph Wh_3 involves one double edge, while the Wh_n graphs with $n\ge 4$ have only single edges. The first nondegenerate case is Wh_4, which is the same graph as K_4. We have given the general structure of $Z(Wh_{n+1},q,v,w)$ in [2], and this determines the structure of $Ph(Wh_{n+1},q,w)$. Reductions for $w=1$ and $w=0$ are given in [19, 20]. For the nondegenerate cases $n\ge 3$, the number of edges is $e(Wh_n)=2(n-1)$. We calculate

$$\begin{aligned} Ph(Wh_{n+1},q,w) &= (q-1)\big[(\lambda_{Wh,+})^n+(\lambda_{Wh,-})^n\big]+(q-1)(q-3)(-1)^n \\ &\quad + w\big[(q-2)^n+(q-2)(-1)^n\big], \end{aligned} \tag{3.30}$$

where

$$\lambda_{Wh,\pm}=\frac{1}{2}\big[q-3\pm\sqrt{A_{Wh}}\,\big] \tag{3.31}$$

with

$$A_{Wh}=(q-3)^2+4w(q-2). \tag{3.32}$$

We note that A_{Wh} is equal to A_1 (given in (4.23)) with q replaced by $q-1$, so that the eigenvalues $\lambda_{Wh,\pm}$ are the same as the eigenvalues $\lambda_{1,0,j}$, $j=1,2$ (given in (4.22)) with q replaced by $q-1$:

$$\lambda_{Wh,\pm}=(\lambda_{1,0,j})_{q\to q-1}, \tag{3.33}$$

where $\pm$ corresponds to $j=1,2$, respectively. The $\lambda_{1,0,j}$, $j=1,2$, enter in $Ph(L_n,q,w)$, given above, and $Ph(C_n,q,w)$, given in (4.21). From these observations, it follows that

$$Ph(Wh_{n+1},q,w)=(q-1)Ph(C_n,q-1,w)+wP(C_n,q-1). \tag{3.34}$$

This relation makes the reductions of $Ph(Wh_{n+1}, q, w)$ for $w = 1$ and $w = 0$ obvious; using $Ph(G, q, 1) = P(G, q)$, one has

$$\begin{aligned} Ph(Wh_{n+1}, q, 1) &= (q-1)P(C_n, q-1) + P(C_n, q-1) \\ &= P(Wh_{n+1}, q) = q\big[(q-2)^n + (q-2)(-1)^n\big] \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} Ph(Wh_{n+1}, q, 0) &= (q-1)P(C_n, q-2) \\ &= P(Wh_{n+1}, q-1) = (q-1)\big[(q-3)^n + (q-3)(-1)^n\big]. \end{aligned} \tag{3.36}$$

Further, from the values of $\chi(Wh_n)$ for odd and even n, it follows that

$$\text{If } n \text{ is odd } Ph(Wh_n, q, w) \text{ contains a factor} \quad (q-1)(q-2). \tag{3.37}$$

and

$$\text{If } n \text{ is even } Ph(Wh_n, q, w) \text{ contains a factor} \quad (q-1)(q-2)(q-3). \tag{3.38}$$

Although Wh_3 differs from $C_3 = K_3$ in having one double edge, (2.16) shows that $Ph(Wh_3, q, w) = Ph(C_3, q, w)$, where $Ph(C_3, q, w)$ is given in (4.27). Furthermore, the graph Wh_4 is the same as K_4, so $Ph(Wh_4, q, w) = Ph(K_4, q, w)$, where $Ph(K_n, q, w)$ was given above in (3.24).

Since the number of edges in the wheel graph $e(Wh_{n+1}) = 2n$, we can apply (2.32) to deduce that

$$\alpha_{Wh_{n+1},n} = -\big(3n + 1 - (n+1)w\big). \tag{3.39}$$

For the following we again assume that $n \geq 3$ so that the Wh_n graph is well-defined. The highest power of w in $Ph(Wh_n, q, w)$ is

$$\deg_w(Ph(Wh_n, q, w)) = \left[\frac{n-1}{2}\right]. \tag{3.40}$$

If $n \geq 6$ is even, say $n = 2m$, then the coefficient of the term in $Ph(Wh_n, q, w)$ of maximal degree, namely the coefficient of the term w^{m-1}, is

$$\beta_{Wh_{2m},m-1} = (2m-1)(q-1)(q-2)^{m-1}(q-3). \tag{3.41}$$

If n is odd, say $n = 2m + 1$, then the coefficient of the term in $Ph(Wh_n, q, w)$ of maximal degree, namely the coefficient of the term w^m, is

$$\beta_{Wh_{2m+1},m} = 2(q-1)(q-2)^m. \tag{3.42}$$

As an illustration, we display $Ph(Wh_5, q, w)$ below:

$$\begin{aligned} Ph(Wh_5, q, w) &= (q-1)(q-2)\big[q^3 - 5(2-w)q^2 \\ &\quad + (2w^2 - 29w + 34)q - (w-1)(4w-39)\big] \\ &= q^5 - (13-5w)q^4 + 2(w^2 - 22w + 33)q^3 - (10w^2 - 140w + 161)q^2 \\ &\quad + (16w^2 - 187w + 185)q - 2(w-1)(4w-39). \end{aligned} \tag{3.43}$$

4 $Ph(G, q, w)$ for Lattice Strip Graphs with Periodic Longitudinal Boundary Conditions

4.1 General Structure

In [1, 2] we have given a general structural formula for $Z(G_s, L_y \times m, BC, q, v, w)$ on strip graphs G_s of width L_y vertices and length L_x, with cyclic (cyc.) or Möbius (Mb) boundary conditions (BC's). For cyclic strips the special case of this structural formula is

$$Ph(G_s, L_y \times m, cyc., q, w) = \sum_{d=0}^{L_y} \tilde{c}^{(d)} \sum_{j=1}^{n_{Ph}(L_y,d)} [\lambda_{G_s,L_y,d,j}(q, w)]^m, \tag{4.1}$$

where $m = L_x$ for strips of the square and triangular lattices and $m = L_x/2$ for strips of the honeycomb lattice. The coefficients $\tilde{c}^{(d)}$ are given by

$$\tilde{c}^{(d)} = \sum_{j=0}^{d} (-1)^j \binom{2d-j}{j} (q-1)^{d-j}. \tag{4.2}$$

The first few of these coefficients are $\tilde{c}^{(0)} = 1$, $\tilde{c}^{(1)} = q - 2$, $\tilde{c}^{(2)} = q^2 - 5q + 5$, etc. For Möbius strips, there is a switching of certain $\tilde{c}^{(d)}$'s as specified in general in [2], using the same methods that we employed in [21], as specified by (2.30)–(2.32) and the $v = -1$ special case of (2.33) of [2].

The numbers $n_{Zh}(L_y, d)$ of λ's corresponding to each $\tilde{c}^{(d)}$ in the general Potts model partition function are reduced for the special case $v = -1$ of interest here. By coloring combinatoric arguments similar to those used in [21] and [2] we determine the $n_{Ph}(L_y, d)$ as follows. The numbers $n_{Ph}(L_y, d)$ are identically zero for $d > L_y$, and

$$n_{Ph}(L_y, L_y) = 1 \tag{4.3}$$

$$n_{Ph}(L_y, L_y - 1) = 2L_y \tag{4.4}$$

$$n_{Ph}(L_y + 1, 0) = n_{Ph}(L_y, 0) + n_{Ph}(L_y, 1) \tag{4.5}$$

and, for $1 \le d \le L_y + 1$,

$$n_{Ph}(L_y + 1, d) = n_{Ph}(L_y + 1, d - 1) + 2n_{Ph}(L_y, d) + n_{Ph}(L_y, d + 1). \tag{4.6}$$

The $n_{Ph}(L_y, d)$ satisfy the identity

$$\sum_{d=0}^{L_y} \tilde{c}^{(d)} n_{Ph}(L_y, d) = P(T_{L_y}, q) = q(q-1)^{L_y - 1}. \tag{4.7}$$

Indeed, one method of calculating the $n_{Ph}(L_y, d)$ is to differentiate this equation L_y times. One thereby obtain $L_y + 1$ linear equations in the $L_y + 1$ unknowns $n_{Ph}(L_y, d)$, $d = 0, 1, \ldots, L_y$; solving these equations yields the results given above. We note that

$$n_{Ph}(L_y, 0) = C_{L_y - 1} + C_{L_y}, \tag{4.8}$$

where C_n is the Catalan number,

$$C_n = \frac{1}{n+1} \binom{2n}{n}. \tag{4.9}$$

(No confusion should result with the use of C_n to mean both Catalan number and the cyclic graph with n vertices, since the context makes clear which is meant.) We recall that the partition function of the zero-field Potts model on cyclic strips of the square lattice (as well as other lattices) has the structure [21, 22]

$$Z(G_s, L_y \times m, cyc., q, v) = \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(L_y,d)} [\lambda_{Z,G_s,L_y,d,j}(q, v)]^m, \tag{4.10}$$

where the coefficients $c^{(d)}$ are given by (4.2) with $q \to q+1$ and

$$n_Z(L_y, d) = \frac{(2d+1)}{(L_y+d+1)} \binom{2L_y}{L_y - d} \tag{4.11}$$

for $0 \le d \le L_y$. For this $h = 0$ case, the total number of distinct eigenvalues that enter in (4.10) for a lattice strip, i.e.,

$$N_{Z,L_y} = \sum_{d=0}^{L_y} n_Z(L_y, d), \tag{4.12}$$

is

$$N_{Z,L_y} = \binom{2L_y}{L_y} \quad \text{for } h = 0. \tag{4.13}$$

We find an interesting relation connecting the numbers $n_{Ph}(L_y, d)$ with the corresponding numbers $n_Z(L_y, d)$ for the zero-field Potts model, namely, for $L_y \ge 2$,

$$n_{Ph}(L_y, d) = n_Z(L_y, d) + n_Z(L_y - 1, d). \tag{4.14}$$

From our determination of the $n_{Ph}(L_y, d)$, we next calculate the total number

$$N_{Ph,L_y} = \sum_{d=0}^{L_y} n_{Ph}(L_y, d). \tag{4.15}$$

We find

$$N_{Ph,L_y} = \binom{2L_y}{L_y} + \binom{2(L_y-1)}{L_y-1}. \tag{4.16}$$

From the relation (4.14) it follows that the total number N_{Ph,L_y} satisfies the relation, for $L_y \ge 2$,

$$N_{Ph,L_y} = N_{Z,L_y} + N_{Z,L_y-1}, \tag{4.17}$$

as is evident in (4.16). We list the $n_{Ph}(L_y, d)$ and N_{Ph,L_y} for $1 \le L_y \le 8$ in Table 1. For purposes of comparison, we include tables of $n_P(L_y, d)$, N_{P,L_y}, and $n_Z(L_y, d)$ for both $h = 0$ and $h \ne 0$ (from [2]) in Appendix A.

Let us denote N_{Z,L_y} for $h \ne 0$ as N_{Zh,L_y} for notational clarity, to distinguish this from $N_Z(L_y, d)$ for $h = 0$. For the Potts model in a nonzero field, we have found [1, 2]

$$N_{Zh,L_y} = \sum_{j=0}^{L_y} \binom{L_y}{j} \binom{2j}{j} \quad \text{for } h \ne 0. \tag{4.18}$$

Table 1 Table of numbers $n_{Ph}(L_y, d)$ and their sums, N_{Ph,L_y} for strips of the lattice Λ (square, triangular, or honeycomb). Blank entries are zero. See text for further discussion

$L_y \setminus d$	0	1	2	3	4	5	6	7	8	N_{Ph,L_y}
1	2	1								3
2	3	4	1							8
3	7	12	6	1						26
4	19	37	25	8	1					90
5	56	118	95	42	10	1				322
6	174	387	350	189	63	12	1			1176
7	561	1298	1276	791	327	88	14	1		4356
8	1859	4433	4641	3185	1533	517	117	16	1	16302

Concerning the relative sizes of N_{P,L_y}, N_{Ph,L_y}, N_{Z,L_y} and N_{Zh,L_y}, we have, for $L_y = 1$, $N_{P,1} = N_{Z,1} = 2 < N_{Ph,1} = N_{Zh,1} = 3$ and the inequality

$$N_{P,L_y} < N_{Z,L_y} < N_{Ph,L_y} < N_{Zh,L_y} \quad \text{for } L_y \geq 2. \tag{4.19}$$

For example, this set of four numbers is (4, 6, 8, 11) and (10, 20, 26, 45) for $L_y = 2$ and $L_y = 3$, respectively.

For large strip width L_y, N_{Ph,L_y} has the same general asymptotic behavior as N_{Z,L_y}:

$$N_{Ph,L_y} \sim \text{const.} \times L_y^{-1/2} 4^{L_y} \quad \text{as } L_y \to \infty. \tag{4.20}$$

4.2 Circuit Graphs C_n

The circuit graph C_n, or equivalently, the 1D lattice with periodic boundary conditions, has chromatic number $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 3$ if $n \geq 3$ is odd. (The case $n = 1$ is a single vertex with a loop, for which there is no proper q-coloring, so $Ph(C_1, q, w)$ vanishes identically.) The polynomial $Ph(C_n, q, w)$ for the circuit graph C_n, can be obtained from the calculations of $Z(C_n, q, v, w)$ [2, 23] by setting $v = -1$. Expressed in our present notation, it is

$$Ph(C_n, q, w) = (\lambda_{1,0,1})^n + (\lambda_{1,0,2})^n + (q-2)(\lambda_{1,1})^n, \tag{4.21}$$

where

$$\lambda_{1,0,j} = \frac{1}{2}\big[q - 2 \pm \sqrt{A_1}\,\big] \tag{4.22}$$

where the $\pm$ sign corresponds to $j = 1, 2$,

$$A_1 = (q-2)^2 + 4(q-1)w \tag{4.23}$$

and

$$\lambda_{1,1} = -1. \tag{4.24}$$

From the values of $\chi(C_n)$ given above, it follows that, in addition to the general factor of $(q-1)$ present for any n, if n is odd, $Ph(C_n, q, w)$ contains a factor of $(q-2)$. Since $e(C_n) = n$, it follows that for $n \geq 3$

$$\alpha_{C_n, n-1} = n(w-2). \tag{4.25}$$

(For $n = 2$, C_2 has a double edge, so one uses (2.32) to obtain $\alpha_{C_2,1} = -(3-2w)$.) We exhibit $Ph(C_n, q, w)$ for $2 \le n \le 5$ below:

$$\begin{aligned} Ph(C_2, q, w) &= Ph(L_2, q, w) = (q-1)[q+2(w-1)] \\ &= q^2 - (3-2w)q + 2(1-w) \end{aligned} \tag{4.26}$$

$$\begin{aligned} Ph(C_3, q, w) &= (q-1)(q-2)\big[q+3(w-1)\big] \\ &= q^3 - 3(2-w)q^2 + (11-9w)q - 6(1-w) \end{aligned} \tag{4.27}$$

$$\begin{aligned} Ph(C_4, q, w) &= (q-1)\big[q^3 + (4w-7)q^2 + (2w^2-16w+17)q - 2(w-1)(w-7)\big] \\ &= q^4 - 4(2-w)q^3 + 2(w^2-10w+12)q^2 - (4w^2-32w+31)q \\ &\quad + 2(w-1)(w-7) \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} Ph(C_5, q, w) &= (q-1)(q-2)\big[q^3 + (5w-7)q^2 + (5w^2-20w+17)q \\ &\quad - 5(w-1)(w-3)\big] \\ &= q^5 - 5(2-w)q^4 + 5(w^2-7w+8)q^3 - 10(2w^2-9w+8)q^2 \\ &\quad + (25w^2-100w+79)q - 10(w-1)(w-3). \end{aligned} \tag{4.29}$$

In the special case of zero-field, $h = 0$, i.e., $w = 1$, $\lambda_{1,0,1} = q-1$ while $\lambda_{1,0,2}$ becomes equal to $\lambda_{1,1}$. Thus, a "transmigration" process occurs in which one of the λ's associated with the coefficient $\tilde{c}^{(d)}$ of degree $d = 0$ becomes equal to, and hence can be grouped with, a λ associated with a coefficient $\tilde{c}^{(d)}$ with a different degree d, here $d = 1$. Hence, one has the reduction

$$Ph(C_n, q, 1) = P(C_n, q) = (q-1)^n + (q-1)(-1)^n. \tag{4.30}$$

For $w = 0$, $\lambda_{1,0,1} = q-2$ while $\lambda_{1,0,2} = 0$, so that $Ph(C_n, q, 0) = Ph(C_n, q-1, 1)$, in agreement with the general relation (1.2).

As an application of our result (2.18) above, it follows that $Ph(C_n, q, w)$ contains the factor $(q-1)$. We note some additional factorization properties and special values of $Ph(C_n, q, w)$:

$$\text{If } n \text{ is odd, then } Ph(C_n, q, w) \text{ contains the factor } (q-2). \tag{4.31}$$

For the $q = 2$ case,

$$Ph(C_n, 2, w) = \big[1 + (-1)^n\big] w^{n/2}. \tag{4.32}$$

The highest power of w in $Ph(C_n, q, w)$ is

$$\deg_w(Ph(C_n, q, w)) = \left[\frac{n}{2}\right], \tag{4.33}$$

where here $[\nu]$ denotes the integral part of ν. This contrasts with the fact that the highest power of w in $Z(C_n, q, v, w)$ for $v \ne -1$ is n. The reason for this is that spin configurations that would yield terms proportional to w^p with $n/2 < p \le n$ for even n and $(n-1)/2 < p \le n$ for odd n are forbidden by the proper q-coloring constraint. For n even, say $n = 2m$, the term in $Ph(C_{2m},q,w)$ of maximal degree in w, namely w^m, has coefficient $2(q-1)^m$. For n odd, say $n=2m+1$ with $m\ge 1$, the term in $Ph(C_{2m+1},q,w)$ of maximal degree in w, namely w^m, has coefficient $(2m+1)(q-1)^m(q-2)$.

If and only if $w=1$, then $Ph(C_n,q,1)=P(C_n,q)$ contains q as a factor. For this zero-field case $w=1$ we also recall that $Ph(C_n,q,1)=P(C_n,q)$ contains $q(q-1)$ as a factor and, furthermore, if $n\ge 3$ is odd, then $Ph(C_n,q)$ also contains $(q-2)$ as a factor.

4.3 $L_y=2$ Cyclic Strip

We denote the cyclic and Möbius strips of the square lattice of width $L_y=2$ and length $L_x=m$ as the ladder graph L_m and the Möbius ladder graph ML_m. For both of these our general structure determination above gives $n_{Ph}(2,0)=3$, $n_{Ph}(2,1)=4$, and $n_{Ph}(2,2)=1$, for a total of $N_{Ph,2}=8$ terms. The weighted coloring polynomial for L_m is

$$Ph(L_m,q,w)=\sum_{d=0}^{2}\tilde{c}^{(d)}\sum_{j=1}^{n_{Ph}(2,d)}(\lambda_{2,d,j})^m, \tag{4.34}$$

where

$$\lambda_{2,0,1}=w(1-q) \tag{4.35}$$

$$\lambda_{2,0,j}=\frac{1}{2}\left[q^2+(w-5)q+7-w\pm\sqrt{A_2}\right],\quad j=2,3 \tag{4.36}$$

$$A_2=q^4+6q^3w+q^2w^2-10q^3-36q^2w-2qw^2+39q^2+72qw+w^2-70q-50w+49 \tag{4.37}$$

$$\lambda_{2,1,j}=-\frac{1}{2}\left[q-2\pm\sqrt{A_3}\right],\quad j=1,2 \tag{4.38}$$

$$A_3=q^2+4(q-1)(w-1) \tag{4.39}$$

$$\lambda_{2,1,j}=-\frac{1}{2}\left[q-4\pm\sqrt{A_4}\right],\quad j=3,4 \tag{4.40}$$

$$A_4=q^2+4q(w-2)+4(4-3w) \tag{4.41}$$

and

$$\lambda_{2,2}=1. \tag{4.42}$$

In (4.40), $j=3$ and $j=4$ apply for the $+$ and $-$ sign choices, respectively, and similarly for the other equations. Results for $Z(L_m,q,v,w)$ are given in [24] and [2]. The weighted chromatic polynomial for the $L_y=2$ Möbius strip of the square lattice is obtained by applying the results of [2].

For $w=1$, these λ's reduce as follows:

$$\lambda_{2,0,1}\to 1-q \tag{4.43}$$

$$\lambda_{2,0,2}\to q^2-3q+3 \tag{4.44}$$

$$\lambda_{2,0,3}\to 3-q \tag{4.45}$$

$$\lambda_{2,1,1}\to 1-q \tag{4.46}$$

$$\lambda_{2,1,2} \to 1 \tag{4.47}$$

$$\lambda_{2,1,3} \to 3-q \tag{4.48}$$

$$\lambda_{2,1,4} \to 1. \tag{4.49}$$

Thus, a transmigration process of λ's occurs here, just as it did for the $L_y = 1$ case; (i) one of the three λ's in the $d = 0$ subspace reduces to the single λ, $q^2 - 3q + 3$, in the $d = 0$ subspace for the chromatic polynomial $P(L_m, q)$, while the other two become equal to the two λ's in the $d = 1$ subspace of $P(L_m, q)$; (ii) two of the four λ's in the $d = 1$ subspace reduce to the two λ's, $3 - q$ and $1 - q$, in this subspace for $P(L_m, q)$, while the other two become equal to the single $\lambda = 1$, in the $d = 2$ subspace for $P(L_m, q)$. Hence, we have the reduction

$$\begin{aligned} Ph(L_m, q, 1) &= (1-q)^m + (q^2-3q+3)^m + (3-q)^m \\ &\quad + \tilde{c}^{(1)}\big[(1-q)^m + 2 + (3-q)^m\big] + \tilde{c}^{(2)} \\ &= (q^2-3q+3)^m + c^{(1)}\big[(3-q)^m + (1-q)^m\big] + c^{(2)}. \end{aligned} \tag{4.50}$$

5 Some Properties of the Zeros of $Ph(G, q, w)$

5.1 Zeros of $Ph(G, q, w)$ in q as Functions of w

Here we discuss the zeros of $Ph(G, q, w)$ in q as a function of w for some illustrative graphs G. Since the maximal degree of $Ph(G, q, w)$ in the variable q is $n(G)$, it has this number of zeros in the variable q. In contrast, as is evident from our explicit calculations above, the maximal degree of $Ph(G, q, w)$ in the variable w depends on details of G. As is true for any polynomial, the positions of the zeros of $Ph(G, q, w)$ are continuous functions of q for fixed w and continuous functions of w for fixed q. As noted above, for any graph G with at least one edge, $Ph(G, q, w)$ contains the factor $(q - 1)$, so it has a fixed zero at $q = 1$. A general statement is that since $Ph(G, q, 1) = P(G, q)$ and $Ph(G, q, 0) = P(G, q - 1)$, it follows that each zero of $Ph(G, q, w)$ shifts horizontally to the right by one unit in the complex q plane if one replaces $w = 1$ by $w = 0$.

One relevant quantity of interest is the maximal real zero of $Ph(G, q, w)$, which we denote $q_{mrz}(G)$. This is related to chromatic number of the graph G, $\chi(G)$, because, by the definition of $q_{mrz}(G)$, $Ph(G, q, w)$ is nonzero for real $q > q_{mrz}(G)$, and it must be positive since for a given G and w, if q is sufficiently large, then $Ph(G, q, w)$ is positive. Hence, in addition to $\chi(G)$, which is fixed for a given G, the quantity $q_{mrz}(G)$ serves as a w-dependent measure of the ability to perform a proper vertex coloring of this graph. As a corollary of the discussion above, a general result is that $q_{mrz}(G)$ shifts one unit to the right as w decreases from 1 to 0, as a consequence of (1.1) and (1.2).

Let us consider some simple examples. We have

$$q_{mrz}(L_1) = 1 - w. \tag{5.1}$$

This increases from 0 to 1 as w decreases from 1 to 0 in the DFCP interval and decreases from 0 through negative values as w increases above 1 in the FCP interval. For L_2,

$$q_{mrz}(L_2) = 2(1 - w). \tag{5.2}$$

This increases from 0 to 2 as w decreases from 1 to 0 in the DFCP interval and decreases from 0 through negative values as w increases from 1 in the FCP interval. This example also illustrates how the multiplicity of zeros can change as a function of w; for $w = 1/2$, $Ph(L_2, q, 1/2)$ has two coincident zeros at $q = 1$.

For L_3, the situation is more complicated. The expression for $Ph(L_3, q, w)$ is given in (3.14). In addition to the fixed zero at $q = 1$, $Ph(L_3, q, w)$ has two other zeros, which occur at the values

$$q_{L3z,\pm} = \frac{1}{2}\big[4 - 3w \pm \sqrt{w(5w-4)}\,\big]. \tag{5.3}$$

For $w = 1$, these reduce to $q = 1$ and $q = 0$ for the $\pm$ signs. As w decreases from 1, the zero at $q = 0$ increases while the zero at $q = 1$ decreases. As w decreases through the value $w = 4/5$, these two zeros meet at $q = 4/5$, and then move off the real axis as a complex-conjugate pair as w decreases further in the interval $0 < w < 4/5$. The magnitudes of the imaginary parts of these complex zeros increase to maximal values as w decreases through the value $w = 2/5$ and then decrease toward zero. As w decreases through the value $w = 0$, these zeros return to the real axis, becoming a double zero at $q = 2$. Thus, for $w = 1$, $q_{L3z,+} = 1$, while for $w = 0$, $q_{L3z,+} = 2$. However, $q_{L3z,+}$ does not increase monotonically from 1 to 2 as w decreases from 1 to 0; instead, it actually decreases from 1 to 4/5 as w decreases from 1 to 4/5, while $q_{L3z,-}$ increases from 0 to 4/5. As w decreases below 4/5, $q_{L3z,\pm}$ form a pair of complex-conjugate roots, as noted. Hence, for w in the DFCP interval, the fixed zero at $q = 1$ is $q_{mrz}(L_3)$. As w passes through the value $w = 0$, $q_{rmz}(L_3)$ jumps discontinuously from the fixed zero at $q = 1$ to $q_{L3z,+} = q_{L3z,-} = 2$.

As w increases above 1 in the FCP interval, $q_{L3z,+}$ decreases monotonically from 1, so that $q_{rmz}(L_1)$ remains the fixed zero at $q = 1$. This example shows that although individual zeros of $Ph(G, q, w)$ in the q plane are continuous functions of w, the maximal real zero $q_{rmz}(G)$ of $Ph(G, q, w)$ is a discontinuous function of w. The reason for the discontinuity in $q_{rmz}(G)$ is the confluence of two complex-conjugate roots that come together and pinch the real axis (at $q = 2$) as w decreases through $w = 0$, abruptly producing a new maximal real root (of multiplicity 2). Thus,

$$\lim_{w\to 0^+} q_{rmz}(L_3) = 1, \tag{5.4}$$

but $q_{rmz}(L_3) = 2$ for $w = 0$.

In the FCP interval $w > 1$, the fixed zero at $q = 1$ remains as $q_{mrz}(L_3)$, since $q_{L3z,+}$ decreases below 1, while $q_{L3z,-}$ decreases below 0. Indeed, the zero $q_{L3z,+}$ has a local maximum at $w = 1$ and decreases monotonically as w increases above 1; $q_{L3z,+}$ passes through 0 as w increases through the value $w = 4$ and behaves asymptotically like $q_{L3z,+} \sim -(1/2)(3-\sqrt{5})w$ as $w \to \infty$. The zero $q_{L3z,-}$ also decreases monotonically from its value of 0 at $w = 1$ through negative values as w increases above 1, and has the asymptotic behavior $q_{L3z,-} \sim -(1/2)(3+\sqrt{5})w$ as $w \to \infty$.

As w decreases through negative values, the double zero $q_{L3z,\pm}$ at $q = 2$ splits apart again. The zero $q_{L3z,+}$ increases monotonically as w decreases through negative values, and grows asymptotically as $q_{L3z,+} \sim (1/2)(3-\sqrt{5})|w|$ as $w \to -\infty$. The other zero, $q_{L3z,-}$ is a non-monotonic function of w; it first decreases below 2, reaching a minimum of $q_{Lz3,-} = 9/5$ for $w = -1/5$ and then increases, passing through the value $q_{Lz3,-} = 2$ again as w decreases through the value $w = -1$, and increasing asymptotically as $q_{L3z,-} \sim (1/2)(3+\sqrt{5})|w|$ as $w \to -\infty$. As these examples show, there is somewhat complicated behavior of the individual zeros as a function of w for even a very simple graph such as L_3, and this behavior is, understandably, more complicated for larger graphs.

In this case and others one can avoid the discontinuous behavior of $q_{rmz}(G)$ by restricting w to the range $w > 0$. Doing this, we have examples from these simple graphs that exhibit a continuous increase of $q_{rmz}(G)$ as w decreases from 1 to 0 and an example in which q_{rmz} is fixed, independent of w in this DFCP interval $0 < w < 1$. For w in the FCP interval $w > 1$ we find cases where $q_{rmz}(G)$ decreases monotonically as w increases and also a case where $q_{rmz}(G)$ is fixed at 1.

Among the $Ph(C_n, q, w)$ polynomials, the case $n = 2$ is the same as $Ph(L_2, q, w)$ and for $n = 3$, there are two w-independent zeros, at $q = 1$ and $q = 2$, while the third occurs at $q = 3(1 - w)$. As w decreases from 1 to 0, this third zero increases from 0 to 3. In the FCP range $w > 1$, this zero decreases from 0 at $w = 1$ to $-\infty$ as $w \to \infty$.

A particularly simple case to discuss is that of complete graphs K_n. For these, as is evident from (3.24), $Ph(K_n, q, w)$ has zeros in q at $q = 1, 2, \ldots, n-1$, and $q = n(1 - w)$.

The zeros of $Ph(G, q, w)$ in q as a function of w for fixed w satisfy certain boundedness properties [25]. For $w = 1$ and $w = 0$, these specialize to the bound for a (usual, unweighted) chromatic polynomial, namely that if q is a zero of $P(G, q)$, then $|q| \le a\Delta_{max}(G)$, where $\Delta_{max}(G)$ denotes the maximal degree of the vertices in G and $a \simeq 7.964$ from [25] (improved in [26]). However, these zeros of $Ph(G, q, w)$ in q are unbounded as $|w| \to \infty$. This is already evident in the simplest case of a single vertex, for which $Ph(L_1, q, w) = q - 1 + w$, with a zero at $q = 1 - w$ with a magnitude that goes to infinity as $|w| \to \infty$. Some insight into the lack of boundedness of the zeros of $Ph(G, q, w)$ for arbitrary w can be gained by examining the behavior of the factor $\prod_{i=1}^{k(G'_i)} (q - 1 + w^{n(G'_i)})$ in $Ph(G, q, w)$. As $w \to \infty$, each of these factors goes to infinity also unless q behaves like $1 - w^{n(G'_i)}$, going to $-\infty$. In principle, one might imagine a cancellation occurring between different $\prod_{i=1}^{k(G'_i} (q - 1 + w^{n(G'_i)})$ factors for different spanning subgraphs $G' \subseteq G$, via different signs of the $(-1)^{e(G')}$ factor in $Ph(G, q, w)$, nevertheless, this makes is understandable why, in the absence of such cancellation, some zero(s) of $Ph(G, q, w)$ have magnitudes $|q| \to \infty$ as $|w| \to \infty$. Note that this behavior cannot simply be attributed to the frustration that occurs when w gets large and positive, because it is also true in the unphysical region for w negative.

Although the positions of the zeros of $Ph(G, q, w)$ in q are continuous functions of w and vice versa, this is not true of the asymptotic locus $\mathcal{B}$. Indeed, we shall show below that for the $n \to \infty$ limit of the circuit graph C_n, as w decreases below one, regardless of how small the magnitude of $1 - w$ is, the part of $\mathcal{B}_q$ that crosses the real q axis on the left jumps discontinuously to the right by one unit, so that this crossing occurs at $q = 1$ instead of at $q = 0$. (In contrast, the right-hand part of $\mathcal{B}_q$ increases above 2 continuously as w decreases below 1.)

5.2 Zeros of $Ph(G, q, w)$ in w as Functions of q

One may also study the zeros of $Ph(G, q, w)$ in w as a function of q. For graphs G containing at least one edge, $Ph(G, 1, w)$ vanishes identically. We therefore take $q \ne 1$, although we shall consider the limit $q \to 1$ below. We again consider some simple examples. $Ph(L_1, q, w) = 0$ for $w = 1 - q$. From (3.13), we find that $Ph(L_2, q, w) = 0$ for $w = 1 - (q/2)$. From (3.14), it follows that $Ph(L_3, q, w) = 0$ at $w = w_{L3z,\pm}(q)$, where

$$w_{L3z,\pm} = \frac{1}{2}\big[5 - 3q \pm \sqrt{(q-1)(5q-9)}\,\big]. \tag{5.5}$$

These roots are real for $q \le 1$ and $q \ge 9/5$, and form a complex-conjugate pair for $1 < q < 9/5$. For $q = 0$, these roots are 1 and 4. As q increases from 0 to 1, $w_{L3z,+}$ decreases monotonically from 4 to 1, while $w_{L3z,-}$ first decreases, reaching a minimum of $4/5$ at $q = 4/5$, and then increases to 1. Thus, at $q = 1$, the roots coalesce to form a double root. As q increases above 1, they split apart to form a complex-conjugate pair, with the magnitude of the imaginary part reaching a maximum at $q = 7/5$, for which $w_{L3z,\pm} = (1/5)(2 \pm \sqrt{5}i)$. As q increases further, these roots move back to the real axis, coming together again at $w = -1/5$ as q increases through the value $9/5$. As q increases further, $w_{L3z,-}$ decreases monotonically, while $w_{L3z,+}$ first increases to the value 0 at $q = 2$ and then decreases. For the complete graph K_n, $Ph(K_n, q, w)$ has a single zero in w at

$$w_{Knz} = 1 - \frac{q}{n}. \tag{5.6}$$

The zeros of $Ph(G, q, w)$ in w as a function of q are not, in general, bounded, even for finite values of q. This is a consequence of the fact that the coefficient of the term in $Ph(G, q, w)$ of highest power of w may vanish as a function of q, in contrast to the fact that the coefficient of the term in $Ph(G, q, w)$ of highest power in q is a w-independent constant (namely, 1) and hence never vanishes. Generically, in the absence of cancellations, a root of an algebraic equation diverges when the coefficient of the term of highest degree vanishes. This is evident in the quadratic equation $aw^2 + bw + c = 0$, where a, b, and c are functions of q with no common factors. Let us denote the set of values of q where $a(q) = 0$ as $\{q_0\}$. One of the roots of this equation, $w = (2a)^{-1}(-b \pm \sqrt{b^2 - 4ac})$, diverges when q approaches one of the values in the set $\{q_0\}$, since $a \to 0$ in this limit. A similar comment applies for algebraic equations of higher degree. Given that we have restricted ourselves, with no loss of generality, to connected graphs G, these all contain at least one edge, except for the case of a single vertex. Hence, for these graphs with at least one edge, $Ph(G, q, w)$ contains the factor $(q - 1)$. It is thus convenient to discuss the reduced coefficients $\bar{\beta}_{G,j}$ defined in (2.36). A zero of $Ph(G, q, w)$ in w has a magnitude that generically diverges when the coefficient of the term of highest degree in w, namely the coefficient $\bar{\beta}_{G,d_w(G)}$, vanishes. This type of divergence can be absent if coefficient(s) $\bar{\beta}_{G,j}$ with $j < d_w(G)$ also vanish sufficiently rapidly as q approaches the value where $\bar{\beta}_{G,d_w(G)}$ vanishes.

We give some simple examples of the divergences in zeros of $Ph(G, q, w)$ in w as a function of q. For the line graph L_4, using our result in (3.15) above, we find that $Ph(L_4, q, w)$ has zeros in w at

$$w_{L4z,1} = 2 - q \tag{5.7}$$

and

$$w_{L4z,2} = -\frac{(q-2)^2}{(3q-4)}. \tag{5.8}$$

As $q - (4/3) \to 0^{\pm}$, $w_{L4z,2} \to \mp\infty$. This divergence is a consequence of the fact that (i) the term in $Ph(L_4, q, w)$ of highest power in w, namely $(q-1)(3q-4)w^2$, has a reduced coefficient $\bar{\beta}_{L_4,2} = 3q - 4$ that vanishes at $q = 4/3$ and (ii) the terms of lower degree in w do not vanish at $q = 4/3$, as is clear from their reduced coefficients $\bar{\beta}_{L_4,1} = 2(q-2)(2q-3)$ and $\bar{\beta}_{L_4,0} = (q-2)^3$.

Another example is provided by the line graph L_6. From (B.4) it follows that $Ph(L_6, q, w)$ has zeros in w at

$$w_{L6z,1} = -\frac{(q-2)^2}{(2q-3)} \tag{5.9}$$

and

$$w_{L6z,j} = \frac{(q-2)[5-4q \pm \sqrt{8q^2-16q+9}]}{4(q-1)}, \quad j = 2, 3, \tag{5.10}$$

where $j = 2$ and $j = 3$ correspond to the $+$ and $-$ signs, respectively. As $q - (3/2) \to 0^{\pm}$, $w_{L6z,1}(q) \to \mp\infty$, and as $q - 1 \to 0^{\pm}$, $w_{L6z,2} \to \mp\infty$ while $w_{L6z,3} \to 1$. The divergences in $w_{L6z,1}$ and $w_{L6z,2}$ are consequences of the fact that (i) the term of highest power in w in $Ph(L_6, q, w)$, namely $2(q-1)^2(2q-3)w^3$, has a reduced coefficient $\bar{\beta}_{L_6,3} = 2(q-1)(2q-3)$ that vanishes at $q = 1$ and $q = 3/2$ and (ii) the terms of subleading degree in w do not vanish at either of these values of q, as is clear from their reduced coefficients $\bar{\beta}_{L_6,2} = (q-2)(10q^2-28q+19)$, $\bar{\beta}_{L_6,1} = 2(3q-4)(q-2)^3$, and $\bar{\beta}_{L_6,0} = (q-2)^5$.

For the Y_5 graph, using our result in (B.2) for $Ph(Y_5, q, w)$, we find that this polynomial has zeros in w at

$$w_{Y5z,1} = 2 - q \tag{5.11}$$

and

$$w_{Y5z,j} = \frac{[-4q^2+13q-11 \pm (q-1)\sqrt{8q^2-28q+25}\,]}{2(2q-3)}, \quad j = 2, 3, \tag{5.12}$$

where $j = 2$ and $j = 3$ correspond to the $+$ and $-$ signs, respectively. As $q - (3/2) \to 0^{\pm}$, $w_{Y5z,3} \to \mp\infty$, while $w_{Y5z,2} \to 1/4$. These divergences can easily be understood from the structure of the reduced coefficients $\bar{\beta}_{Y_5,j}$ for the various powers of w in $Ph(Y_5, q, w)$, as discussed in general above.

As a last example, for the H_6 graph, using our calculation in (B.7), we find that $Ph(H_6, q, w)$ has a double zero at

$$w_{H6,1} = 2 - q \tag{5.13}$$

and two other zeros at

$$w_{H6,j} = \frac{[-2q^2+6q-5 \pm (q-1)\sqrt{3q^2-10q+9}]}{q-2}, \quad j = 2, 3, \tag{5.14}$$

where $j = 2, 3$ for the $\pm$ signs, respectively, as before. As $q - 2 \to 0^{\pm}$, $w_{H6z,3} \to \mp\infty$, while $w_{H6z,2} \to 0$.

As these examples show, the zeros of $Ph(G, q, w)$ in w can be unbounded as functions of q. The divergences in these zeros that we have found occur at values of $q \geq 1$ (including the integers $q = 1$ and $q = 2$). It is of interest to determine a region in the q plane for which the zeros of $Ph(G, q, w)$ in w are bounded, and we are studying this problem. Our results show that such a region would have to exclude the real interval $1 \leq q \leq 2$.

6 Quantities Defined in the Limit $n(G) \to \infty$

6.1 Φ Function

From the chromatic polynomial $P(G, q) \equiv Ph(G, q, 1)$, one defines a configurational degeneracy, which is the ground-state degeneracy, when viewing $P(G, q)$ as the partition function of the zero-temperature Potts antiferromagnet,

$$W(\{G\}, q) = \lim_{n\to\infty} P(G, q)^{1/n}, \tag{6.1}$$

where $n = n(G)$ and we use the symbol $\{G\}$ to denote the limit $n \to \infty$ for a given family of graphs (and the symbol W should not be confused with the variable w). In the present context, this $n \to \infty$ limit corresponds to the limit of infinite length for a strip graph of fixed width and some prescribed boundary conditions. The associated configurational entropy per vertex (ground-state entropy per site of the Potts antiferromagnet) for $\{G\}$ is

$$S = k_B \ln W. \tag{6.2}$$

The third law of thermodynamics states that the entropy per site S goes to zero as the temperature goes to zero. However, there are a number of exceptions to this law. Elementary lower bounds on W are $W \ge (q-1)^{1/2}$ on a bipartite graph and $W \ge (q-2)^{1/3}$ on a tripartite graph. Hence $W > 1$, and $S > 0$ for (i) $q > 1$ and (ii) $q > 2$ on the $n \to \infty$ limit of a (i) bipartite and (ii) tripartite graph, respectively. In each of these cases, the third law is violated. A well-known violation in nature is water ice, which exhibits ground-state entropy [27].

In the present case with a nonzero external magnetic field $H \neq 0$, we define an analogous quantity

$$\Phi(\{G\}, q, w) = \lim_{n(G)\to\infty} Ph(G, q, w)^{1/n}. \tag{6.3}$$

As before (cf. (1.9) of [28] and (2.8) of [29]), one must take account of a noncommutativity of limits, namely the fact that for certain special values of q, denoted $\{q_s\}$, the limits $n \to \infty$ and $q \to q_s$ do not commute:

$$\lim_{n\to\infty} \lim_{q\to q_s} Ph(G, q, w)^{1/n} \neq \lim_{q\to q_s} \lim_{n\to\infty} Ph(G, q, w)^{1/n}. \tag{6.4}$$

Because of these noncommutativities, the formal definition (6.3) is, in general, insufficient to define Φ at these special points; it is necessary to specify the order of the limits that one uses in (6.4). This noncommutativity also affects the resultant accumulation sets $\mathcal{B}$. We have discussed this in detail before in the case of the chromatic polynomial [28] and zero-field Potts model partition function [29]. Modulo this subtlety, it follows from (1.1) and (1.2) that

$$\Phi(\{G\}, q, 1) = W(\{G\}, q) \tag{6.5}$$

and

$$\Phi(\{G\}, q, 0) = W(\{G\}, q-1). \tag{6.6}$$

6.2 Accumulation Locus $\mathcal{B}$ for Strip Graphs and Φ Function

For a n-vertex graph G in a recursive family of graphs such as strip graphs, as $n \to \infty$, a subset of the zeros of $Ph(G, q, w)$ merge to form a locus $\mathcal{B}$. For fixed w, this is a locus $\mathcal{B}_q$ in the q plane, while for fixed q, it is a locus $\mathcal{B}_w$ in the w plane. We define a dominant (maximal) eigenvalue λ_{max} as an eigenvalue whose magnitude $|\lambda_{max}|$ is larger than or equal to the magnitudes of all other eigenvalues. From (2.30), it follows that the zeros of $Ph(G, q, w)$ can occur either as an isolated zero of a single dominant eigenvalue or where two dominant eigenvalues are equal in magnitude. The continuous accumulation set of the zeros of $Ph(G, q, w)$ in a given variable, denoted $\mathcal{B}$, is given generically by the solution set of the condition of equality of dominant eigenvalues. For real w, the coefficients of the terms in $Ph(G, q, w)$ are also real (actually integers, although this is not used here), and consequently, the set of zeros of $P(G, q, w)$ in the complex q plane is invariant under complex conjugation $q \to q^*$. *A fortiori*, the locus $\mathcal{B}_q$ is also invariant under this complex conjugation. By the same logic, for real q, the set of zeros of $Ph(G,q,w)$ in the complex w plane is invariant under the complex conjugation $w \to w^*$, and so is their accumulation set $\mathcal{B}_w$. Because the λ's are the same for lattice strip graphs with cyclic and Möbius boundary conditions, it follows that in the $L_x \to \infty$ limit, the loci $\mathcal{B}$ are also the same. With regard to the zeros of $Ph(G,q,w)$ in w for fixed q, we note that these are to be contrasted with the zeros of the Potts model partition function $Z(G,q,v,w)$ in w for fixed q and $v \neq -1$, which have studied previously in many works.

For the strip graphs of width L_y considered here,

$$\Phi(\{G\},q,w) = (\lambda_{max})^{1/L_y}. \tag{6.7}$$

As one moves across a locus $\mathcal{B}$, there is thus, generically, a switching of dominant eigenvalues and an associated non-analyticity in Φ. From (1.2) we have

$$\Phi(\{G\},q,0) = \Phi(\{G\},q-1,1). \tag{6.8}$$

7 Φ Function and Accumulation Locus $\mathcal{B}$ for Line Graphs

7.1 $\mathcal{B}_q$

Only $\lambda_{1,0,j}$, $j=1,2$ contribute to $Ph(L_n,q,w)$. For $w=1$ we encounter the noncommutativity of (6.4). If we first set $w=1$ and then vary n, we can use the fact that $Ph(L_n,q,1) = P(L_n,q) = q(q-1)^{n-1}$, so that aside from the single zero at $q=0$, the zeros accumulate at $q=1$ and $\mathcal{B}_q$ degenerates to this single point. If we choose the other order of limits, first taking $n \to \infty$ and then $w \to 1$, then the locus $\mathcal{B}_q$ is the solution to the equation $|\lambda_{1,0,1}| = |\lambda_{1,0,2}|$. This equation is always satisfied if $q=2$, so this point is on $\mathcal{B}_q$, which forms a complex-conjugate arc with endpoints where $A_1=0$, where A_1 was given in (4.23). As $w \to 1$, these endpoints come together at $q=0$.

For $w=0$, $Ph(L_n,q,0) = P(L_n,q-1) = (q-1)(q-2)^{n-1}$, so the locus $\mathcal{B}_q$ degenerates to the single point at $q=2$. We proceed to consider $w \neq 0,1$. Here, the equation $|\lambda_{1,0,1}| = |\lambda_{1,0,2}|$ determines the locus $\mathcal{B}_q$. The DFCP interval $0 \le w \le 1$ is of particular interest, since for this interval the weighted chromatic polynomial $Ph(G,q,w)$ interpolates between two chromatic polynomials; $Ph(G,q,1) = P(G,q)$ and $Ph(G,q,0) = P(G,q-1)$. We thus study $\mathcal{B}_q$ for this interval first. In this DFCP interval, $\mathcal{B}$ forms a (self-conjugate) arc passing through $q=2$ and ending at the points where $A_1=0$, namely $q_{e,j}$, $j=1,2$,

$$q_{e,j} = 2\big[1-w \pm \sqrt{w(w-1)}\,\big], \quad j=1,2, \tag{7.1}$$

where $j=1,2$ correspond to the $\pm$ signs, respectively. Here the square root in (7.1) is pure imaginary, so $q_{e,1} = q_{e,2}^*$ form a complex-conjugate pair. As w decreases below 1, these arc endpoints move away from the real axis near $q=0$. They reach their maximal distance from the real axis at $w=1/2$, where $q_{e,j} = 1 \pm i$, and as w decreases from $1/2$ to 0, these arc endpoints come back toward this axis, finally reaching it at $q=2$.

In the FCP interval $w>1$, the locus $\mathcal{B}_q$ is a line segment whose right and left ends occur at $q_{e,1}$ and $q_{e,2}$, respectively. As w increases above $w=1$, the line segment extends outward from the point $q=0$. Some illustrative sets of values for these endpoints are (i) for $w=1.2$, $q_{e,1} \simeq 0.580$ and $q_{e,2} \simeq -1.380$, (ii) for $w=2$, $q_{e,1} \simeq 0.828$ and $q_{e,2} \simeq -4.828$, and (iii) for

$w = 10$, $q_{e,1} = 0.974$ and $q_{e,2} = -36.974$. As $w \to \infty$, the right end of this line segment occurs asymptotically at

$$q_{e,1} = 1 - \frac{1}{4w} - \frac{1}{8w^2} - O\left(\frac{1}{w^3}\right), \tag{7.2}$$

while the left end occurs at approximately

$$q_{e,2} = -4w + 3 + \frac{1}{4w} + O\left(\frac{1}{w^2}\right). \tag{7.3}$$

As discussed above, the ranges of w for these weighted coloring problems are $0 \le w < 1$ and $w > 1$, respectively. We may also consider an extension of this range of values of w in which w becomes negative, although negative values of w do not correspond to a weighted graph coloring problem. As w decreases from 0 through negative values, $\mathcal{B}_q$ forms a line segment that extends outward from the point $q = 2$, with left and right ends at $q_{e,1}$ and $q_{e,2}$, respectively. As $w \to -\infty$, the left end approaches $q = 1$ from above, as $q_{e,1} \simeq 1 + (4|w|)^{-1} + O(w^{-2})$, while the right end goes to infinity, as $q_{e,2} \simeq 4|w| + 3 - (4|w|)^{-1}$. This locus $\mathcal{B}_q$ does not separate the complex q into any separate regions. The dominant λ is $\lambda_{1,0,1}$ and, denoting the formal limit of this family of line graphs as $\lim_{n\to\infty} L_n = \{L\}$, the resultant Φ function is

$$\Phi(\{L\}, q, w) = \lambda_{1,0,1} = \frac{1}{2}\left[q - 2 + \sqrt{(q-2)^2 + 4(q-1)w}\,\right]. \tag{7.4}$$

Here it is understood that one takes account of the branch cut associated with the branch point singularity in the square root, so that at large negative q, $|\lambda_{1,0,1}| \sim -q$. For $w \simeq 1$, this has the Taylor series expansion

$$\Phi(\{L\}, q, w) = q - 1 + \frac{[(q-1)(w-1)]}{q} - \frac{[(q-1)(w-1)]^2}{q^3} + \frac{2[(q-1)(w-1)]^3}{q^5} - O\big((w-1)^4\big) \quad \text{as } w \to 1. \tag{7.5}$$

For $w \simeq 0$, the Taylor series expansion for $\Phi(\{L\}, q, w)$ is

$$\Phi(\{L\}, q, w) = (q-2)\big[1 + z - z^2 + 2z^3 - O(z^4)\big] \quad \text{as } w \to 0, \tag{7.6}$$

where here we use the compact notation

$$z = \frac{(q-1)w}{(q-2)^2}. \tag{7.7}$$

As $|w| \to \infty$, $\Phi(\{L\}, q, w)$ behaves asymptotically as

$$\Phi(\{L\}, q, w) \sim \sqrt{(q-1)w}\left[1 + \frac{q-2}{2\sqrt{(q-1)w}} + O\left(\frac{1}{w}\right)\right] \quad \text{as } |w| \to \infty. \tag{7.8}$$

As $q \to \infty$, $\Phi(\{L\}, q, w)$ behaves asymptotically as

$$\Phi(\{L\}, q, w) \sim q - 2 + w - \frac{w(w-1)}{q} + \frac{2w(w-1)^2}{q^2} + O\left(\frac{1}{q^3}\right) \quad \text{as } q \to \infty. \tag{7.9}$$

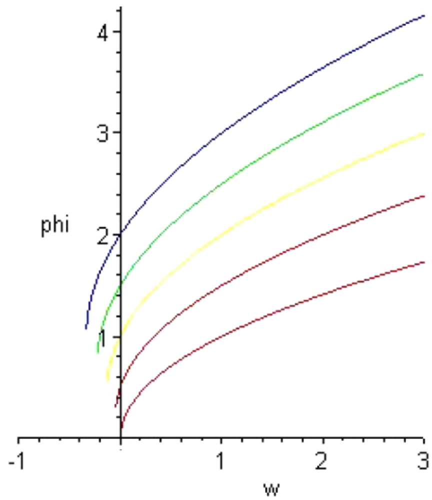


Fig. 2 Plot of $\Phi(\{L\}, q, w)$ as a function of w for the following values of q, from *bottom* to *top*: $q =$ (a) 2, (b) 2.5, (c) 3, (d) 3.5, (e) 4

In Fig. 2 we show plots of $\Phi(\{L\}, q, w)$ as a function of w for some representative values of q. As is evident from Fig. 2, for $q > 1$, $\Phi(\{L\}, q, w)$ is a monotonically increasing function of w in the DFCP range $0 < w < 1$. This reflects the fact that it is easier to carry out a proper q-coloring of the line graph as the weighting factor w increases from 0 to 1. It is also easy to understand why $\Phi(\{L\}, q, w)$ is, for $q > 1$, a monotonically increasing function of w in the FCP interval of real $w > 1$, with the leading asymptotic form given in (7.8), $\Phi(\{L\}, q, w) \sim \sqrt{(q-1)w}$. To see this, let us start with finite n and maximize the number of vertices assigned the color 1, in order to maximize the power of w in $\Phi(L_n, q, w)$. Since the graph L_n is bipartite, we can start at, say, the left end of the line graph and assign the corresponding vertex, denoted vertex $i = 1$ with this color 1. Then the next vertex to the right, $i = 2$, cannot be assigned this color, but there are $q - 1$ possibilities for its color. The vertices $i = 3,\ 5$, and so forth for all odd-numbered vertices, are similarly assigned the color 1. Each of the even-numbered vertices can independently be assigned any of $q - 1$ colors. Taking into account all of these possible color assignments (or equivalently, spin configurations, in the statistical mechanics context), it follows that, if n is even, then the dominant term in $\Phi(L_n, q, w)$ as $w \to \infty$ for $q > 1$ is $Ph(L_n, q, w) = [(q-1)w]^{n/2}$ and if n is odd, then this dominant term is $Ph(L_n, q, w) = (q-1)^{(n-1)/2} w^{(n+1)/2}$. In either case, for $q > 1$, as $w \to \infty$, if one takes $n \to \infty$ and calculates $\Phi(\{L\}, q, w)$, one obtains, as the leading asymptotic expression, $\Phi(\{L\}, q, w) \sim \sqrt{(q-1)w}$.

Although negative w is not associated with any coloring problem, we also show in Fig. 2 the extensions of the curves into the negative-w region. The function A_1 in the square root of $\Phi(\{L\}, q, w)$ becomes negative, and hence $\Phi(\{L\}, q, w)$ becomes complex, for $w < w_z(q)$, where

$$w_z(q) = -\frac{(q-2)^2}{4(q-1)}. \tag{7.10}$$

For example, $w_z(2) = 0$, $w_z(3) = -1/8$, and $w_z(4) = -1/3$. We only plot each curve for $w > w_z(q)$. The values at $w = 0$ and $w = 1$ are $\Phi(\{L\}, 0, w) = q - 2$ and $\Phi(\{L\}, 1, w) = q - 1$.

In Figs. 3 and 4 we show $\Phi(\{L\}, q, w)$ as a function of q for the range relevant for coloring, namely $q \geq 1$, and a set of w values in the DFCP interval $0 < w < 1$ and the FCP

Fig. 3 Plot of $\Phi(\{L\}, q, w)$ as a function of $q \geq 1$ for the following values of w: $w =$ (a) 0.1, (b) 0.2, (c) 0.5, (d) 0.8, (e) 0.9. The plot also shows $Ph(\{L\}, q, 0) = q - 2$ and $Ph(\{L\}, q, 1) = q - 1$. The curves for the w values (a)–(e) are arranged from *bottom* to *top*, to the right of $q = 1$, between these lines

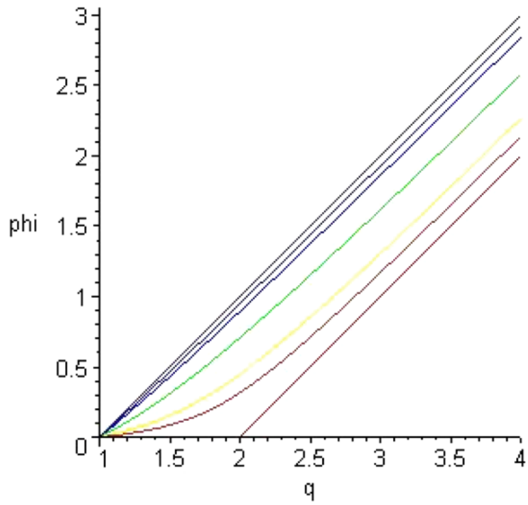
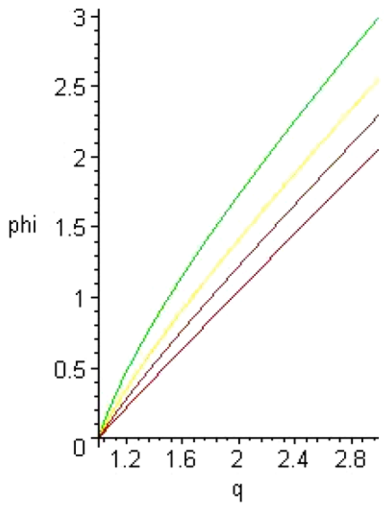


Fig. 4 Plot of $\Phi(\{L\}, q, w)$ as a function of $q \geq 1$ for the following values of w in the FCP interval: $w =$ (a) 1.1, (b) 1.5, (c) 2, (d) 3. The curves for the w values (a)–(d) are arranged from *bottom* to *top*



interval $w > 1$, respectively. One sees that for fixed $q > 1$, $\Phi(\{L\}, q, w)$ is a monotonically increasing function of w for $w > 0$. Given its definition as the $1/n$'thd power of the weighted chromatic polynomial $Ph(G, q, w)$ as $n \to \infty$ and the fact that $Ph(G, q, w)$ provides a measure of the ease of carrying out a weighted proper q-coloring of the graph G, it follows that where $\Phi(\{G\}, q, w)$ is meaningful for such colorings, it is non-negative. Hence the continuation of the line that passes through $q = 2$ to negative values of Φ is not relevant for graph coloring.

These calculations with line graphs show a number of general features of the weighted chromatic polynomial $Ph(G, q, w)$ and the associated limiting function $\Phi(G, q, w)$. They suggest the following conjectured generalization for weighted colorings of families of strip graphs G of regular lattices Λ: in the $n \to \infty$ limit of G, denoted $\{G\}$. Let the chromatic number of the lattice Λ be indicated as $\chi(\Lambda)$. For a bipartite lattice such as a line graph L_n, a circuit graph C_n with even n, or a square, cubic, or body-centered cubic lattice, $\chi(\Lambda_{bip.}) = 2$.

For a triangular lattice, $\chi(tri) = 3$, etc. Assume $q \geq \chi(\Lambda)$. Then (i) for fixed $w > 0$, $\Phi(\{G\}, q, w)$ is a monotonically increasing function of q and (ii) for fixed q, $\Phi(\{G\}, q, w)$ is a monotonically increasing function of w for $w > 0$. The generalization (i) is understandable since $\Phi(G, q, w)$ is a measure of the number of weighted proper q-coloring of G, and this should increase if there are more colors, i.e., if q increases. For the DFCP interval $0 < w < 1$, the generalization (ii) follows because increasing w in this interval removes the penalty factor for coloring a vertex with one of the colors and hence clearly makes it easier to perform a proper q-coloring of the vertices of G. In the FCP interval $w > 1$, the generalization (ii) is rendered plausible because for sufficiently large q, one can analyze the dominant terms contributing to $\Phi(G, q, w)$, and these arise from maximizing the number of vertices that can be assigned the color 1, subject to the proper q-coloring condition, and then enumerating the possible color assignments for the other vertices. We remark that for these monotonicity properties, it is important that $q \geq \chi(G)$. As an example of how the behavior differs when $q < \chi(G)$, consider the weighted chromatic polynomial of the complete graph, K_n, given in (3.24). Recall that $\chi(K_n) = n$. Let us take $n = 4$ for definiteness, so that $Ph(K_4, q, w) = (q-1)(q-2)(q-3)[q + 4(w-1)]$. This is a monotonically increasing function of q if $q > 3$, but not for smaller values of q. Moreover, say we keep w arbitrary but choose the illustrative value $q = 5/2$, whence $Ph(K_4, 5/2, w) = (3/16)(3 - 8w)$. This is not an increasing function of w in the DFCP or FCP intervals.

7.2 $\mathcal{B}_w$

One can also study $\mathcal{B}_w$ as a function of q. We find that $\mathcal{B}_w$ is the semi-infinite real line segment

$$\mathcal{B}_w: \quad w < w_z(q) \quad \text{for } \{L\}, \tag{7.11}$$

where $w_z(q)$ was given in (7.10). For the range of q relevant for weighted graph coloring, namely $q > 1$, $w_z(q) \leq 0$.

8 Φ Function and Accumulation Locus $\mathcal{B}_q$ for Circuit Graphs

Here we discuss the accumulation set $\mathcal{B}$ of the zeros of $Ph(G, q, w)$ as $n \to \infty$ for the family of circuit graphs, C_n (this limit is denoted $\{C\}$). The results depend on w, so we discuss various intervals of w in turn.

8.1 $w = 1$

We begin by briefly reviewing the results for the case unweighted case $w = 1$, i.e., for the chromatic polynomial $P(C_n, q)$, given in (4.30). The resultant locus $\mathcal{B}_q$ in the limit $n \to \infty$ is the unit circle

$$\mathcal{B}_q: \quad |q-1| = 1 \quad \text{for } w = 1. \tag{8.1}$$

This crosses the real axis at the points

$$q_{cr,1} = 0, \qquad q_{cr,2} = 2 \quad \text{for } w = 1. \tag{8.2}$$

Following our earlier notation [28, 30–32], we denote the maximal point at which $\mathcal{B}_q$ intersects the real q axis for the $n \to \infty$ limit of a given family of graphs, $\{G\}$, as $q_c(\{G\})$. Thus, here, $q_c = 2$.

Indeed, all of the complex zeros lie exactly on this unit circle [33]. The boundary $\mathcal{B}_q$ separates the q plane into two regions, in which $W(q)$ has different analytic forms. Outside of the circle $|q-1|=1$, the dominant λ is $\lambda_{1,0,1}=q-1$, while inside of this circle, the dominant λ is $\lambda_{1,1}=-1$, so

$$\begin{aligned} W(q) &= q-1 \quad \text{for } |q-1|>1 \\ |W(q)| &= 1 \quad \text{for } |q-1|<1. \end{aligned} \tag{8.3}$$

8.2 $0 \le w < 1$

As w decreases below 1 in the interval $0 \le w < 1$, the boundary $\mathcal{B}_q$ continues to be a simple closed curve separating the q plane into two regions. However, there is a discontinuous change in the form of this boundary. On the left, the point at which the locus $\mathcal{B}_q$ crosses the real q axis jumps from $q=0$ for $w=1$ to

$$q_{cr,1}=1 \quad \text{for } 0 \le w < 1. \tag{8.4}$$

Associated with this, the left part of the boundary $\mathcal{B}_q$ changes discontinuously from a section of a circle to an involuted cusp with its tip at $q=1$. (There may also be discrete zero(s) to the left of $q=1$.) Thus, as one moves along the curve forming $\mathcal{B}_q$ upward from the point $q=1$ where it intersects the real axis in the cusp, this curve moves to the upper left, finally curving around to go upward, and then over to the right. The behavior of the boundary $\mathcal{B}_q$ on the right side is continuous as w deviates from 1; this part of $\mathcal{B}_q$ crosses the real axis at

$$q_{cr,2}=q_c=\frac{w+3}{w+1} \quad \text{for } \{G\}=\{C\}. \tag{8.5}$$

This point q_c increases monotonically from $q_c=2$ for $w=1$ to $q_c=3$ as w decreases from 1 to 0. We denote as R_1 the region that includes the real interval $q>q_c$ and the part of the complex q plane analytically connected to it, which is the region outside of the closed curve formed by $\mathcal{B}_q$, For w only slightly less than unity, $\mathcal{B}_q$ has the form of a lima bean, with its concave part facing left and its convex part facing right. As w decreases through this interval $0 \le w < 1$, the bulbous parts of $\mathcal{B}_q$ on the upper left and lower left disappear, and eventually, as w decreases toward 0, the locus $\mathcal{B}_q$ becomes the circular locus (8.1) with q replaced by $q-1$ (in accordance with (1.2)), i.e., unit circle whose center is shifted horizontally by one unit to the right in the q plane:

$$\mathcal{B}_q: \quad |q-2|=1 \quad \text{for } w=0. \tag{8.6}$$

In region R_1,

$$\Phi(\{C\},q,w)=\Phi(\{L\},q,w) \quad \text{for } q \in R_1, \tag{8.7}$$

where $\Phi(\{L\},q,w)$ was given above in (7.4). In region R_2 forming the interior of the closed curve $\mathcal{B}_q$,

$$|\Phi(\{C\},q,w)|=1 \quad \text{for } q \in R_2. \tag{8.8}$$

Thus, a plot of $\Phi(\{C\},q,w)$ as a function of q for $w \in (0,1)$ is similar to the plot of $\Phi(\{L\},q,w)$ given in Fig. 3 but with the difference that the curve extends only down to the value $q=q_c$ given in (8.5), where $\Phi(\{L\},q_c,w)=\Phi(\{C\},q_c,w)=1$, and for $1 \le q \le q_c$, $\Phi(\{C\},q,w)$ has unit magnitude in region R_2, while $\Phi(\{L\},q,w)$ continues downward, reaching zero as $q \to 1$.

8.2.1 $w > 1$

For the range $w > 1$, the locus $\mathcal{B}_q$ contains a line segment on the real axis, whose left end occurs at $q_{e,2}$ (cf. (7.1)). This is a solution of the condition that $A_1 = 0$, where A_1 is the function in the square root in (4.23). Along this line segment, $A_1 < 0$, so that the square root in (4.22) is pure imaginary; hence, the $\lambda_{1,0,j}$ with $j = 1, 2$ are complex conjugates of each other. They are also larger than $|\lambda_{1,1}| = 1$ and hence this line segment is on $\mathcal{B}_q$. As one moves to the right along the real axis on this line segment, when one comes to the intermediate point q_{int}, given by

$$q_{int} = 1 - \frac{1}{w}, \tag{8.9}$$

the equal magnitudes of $\lambda_{1,0,j}$, $j = 1, 2$ decrease through the value 1, so this is the point at which this line segment on $\mathcal{B}_1$ terminates. For the present range $w > 1$, the point $q = 0$ is always on this line segment, since

$$\lambda_{1,0,j}(q = 0) = -1 \pm i\sqrt{w - 1}, \quad j = 1, 2, \tag{8.10}$$

and these are dominant over $\lambda_{1,1} = -1$, since $|\lambda_{1,0,j}| = 1 + |w - 1|$ for $j = 1, 2$. In the real interval

$$q_{int} \le q \le 1, \tag{8.11}$$

and in the region of the complex q plane analytically connected with it, $\lambda_{1,1} = -1$ is dominant. We denote this region as R_3, and

$$|\Phi(\{C\}, q, w)| = 1 \quad \text{for } q \in R_3. \tag{8.12}$$

Thus, although the square root in the $\lambda_{1,0,j}$ is imaginary for the full interval $q_{e,2} \le q \le q_{e,1}$, this does not affect $\mathcal{B}_q$ for $q > q_{int}$.

At $q = 1$, $\lambda_{1,0,2}$ becomes degenerate with $\lambda_{1,1}$, so $\mathcal{B}_q$ crosses the real q axis at this point. The eigenvalue $\lambda_{1,0,2}$ is dominant in the interval

$$1 \le q \le 2 \tag{8.13}$$

and the region of the q plane analytically connected with it. We denote this region as R_2, and obtain

$$\Phi(\{C\}, q, w) = \frac{1}{2}\left[q - 2 - \sqrt{(q-2)^2 + 4(q-1)w}\,\right] \quad \text{for } q \in R_2. \tag{8.14}$$

At the point

$$q = q_c = 2 \tag{8.15}$$

$\lambda_{1,0,1} = -\lambda_{1,0,2} = \sqrt{w}$, and both are dominant over $\lambda_{1,1} = -1$, since $w > 1$. Hence, $\mathcal{B}_q$ crosses the real q axis again at this point, and this is the maximal real value of q where $\mathcal{B}_q$ crosses the real axis. We denote the real interval $q > 2$ and the region of the complex q plane analytically connected with it as region R_1. In this region, since $\lambda_{1,0,1}$ is dominant,

$$\Phi(\{C\}, q, w) = \frac{1}{2}\left[q - 2 + \sqrt{(q-2)^2 + 4(q-1)w}\,\right] \quad \text{for } q \in R_1. \tag{8.16}$$

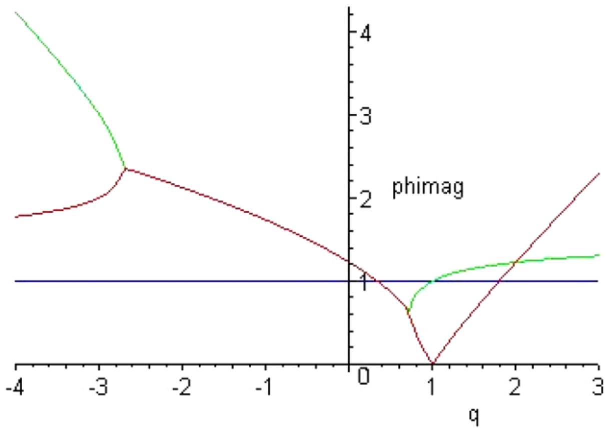


Fig. 5 Plot of (a) $|\lambda_{1,0,1}|$, (b) $|\lambda_{1,0,2}|$, and (c) $|\lambda_{1,1}| = 1$ for $Ph(C_n, q, w)$ with the illustrative value $w = 3/2$. The values of these magnitudes and the corresponding order of the curves from *top* to *bottom* are (i) $a > b > c$ for $q > 2$; (ii) $b > a > c$ for $q \in (1, 2)$, (iii) $c > b > a$ for $q_{int} < q < 1$, (iv) $a = b > c$ for $q_{e,2} < q < q_{int}$, (v) $a > b > c$ for $q < q_{e,2}$. For $w = 3/2$, $q_{int} = 1/3$ and $q_{e,2} \simeq -2.732$. See text for further discussion

Thus, for $w > 1$, the locus $\mathcal{B}_q$ for the $n \to \infty$ limit of $Ph(C_n, q, w)$ separates the q plane into three regions.

In Fig. 5 we plot $|\lambda_{1,0,1}|$, $|\lambda_{1,0,2}|$, and $|\lambda_{1,1}| = 1$ as functions of q for the illustrative value $w = 3/2$. For this value, $q_{int} = 1/3$ and $q_{e,2} = -(1 + \sqrt{3}\,) \simeq -2.732$. As before, the λ of dominant magnitude determines $\Phi(\{C\}, q, w)$. For $q > 2$, $\lambda_{1,0,1}$ is dominant; for $1 \le q \le 2$, $\lambda_{1,0,2}$ is dominant; for $1/3 \le q \le 1$, $\lambda_{1,1}$ is dominant, and for $q < q_{e,2}$, $\lambda_{1,0,1}$ is again dominant (taking into account the branch cut in the definition of the sign of the square root). On the interval $q_{e,2} \le q \le q_{int}$, $|\lambda_{1,0,1}|$ and $|\lambda_{1,0,2}|$ are equal in magnitude and dominant. This plot shows the intersections of the curves (or line) where there is a degeneracy of dominant λ's at $q = 2$, $q = 1$, and along the interval $q_{e,2} \le q_{int}$.

8.3 $w < 0$

As w decreases below 0 through the range $-1/3 < w < 0$, $\mathcal{B}_q$ forms a closed curve that crosses the real axis at the fixed point $q = q_{cr,1} = 1$ and at the w-dependent point $q = q_{cr,2}$, which increases as w becomes more negative. The shape of $\mathcal{B}_q$ changes as w becomes more negative in this interval, becoming a teardrop with its broadly rounded end on the left and its sharper end on the right. As w decreases through the value $w = -1/3$, a line segment appears, and for $w < -1/3$, the rightmost part of the locus $\mathcal{B}$ is comprised by this line segment, which extends from $q = q_{int}$ in (8.9) to $q_{e,1}$ in (7.1). Thus, for $w < -1/3$, the locus $\mathcal{B}$ consists of a teardrop-shaped curve crossing the real axis on the left at $q = 1$ and on the right at $q = q_{int}$ (with the bulbous part of the teardrop on the left and the sharp part on the right), together with a line segment extending over the interval $q_{int} \le q \le q_{e,1}$. At $w = -1/3$,

$$(\lambda_{1,0,j})_{w=-1/3} = \frac{1}{2}\Big[(q-2) \pm \sqrt{(q-4)(q-(4/3))}\,\Big]. \tag{8.17}$$

For this value $w = -1/3$, and $q = 4$, the pair $\lambda_{1,0,j}$, $j = 1, 2$, are equal to each other and are also equal to the magnitude $|\lambda_{1,1}| = 1$. As $w \to -\infty$, $q_{int} \to 1^+$, so the teardrop curve contracts to a point at $q = 1$ while the right-hand endpoint of the line segment at $q_{e,1}$ approaches infinity like $q_{e,1} \sim 4|w|$.

One can also calculate the loci $\mathcal{B}_q$ for the $n \to \infty$ limits of other families of graphs. However, our discussion for the $n \to \infty$ limits of line graphs L_n and circuit graphs C_n already exhibit a number of salient features of these loci.

9 Locus $\mathcal{B}_w$ for Circuit Graphs

Here we discuss the locus $\mathcal{B}_w$ as a function of q for the $n \to \infty$ limit of the circuit graph C_n. First, we note that if $q = 1$ or $q = 2$, then we encounter the noncommutativity (6.4). For $q = 1$, this is evident from (2.18), according to which if we set $q = 1$ first and then take $n \to \infty$, the problem is trivial, since $Ph(C_n, 1, w)$ vanishes identically. For $q = 2$, the noncommutativity is evident from our general result in (4.32), because the coefficient of the $(\lambda_{1,1})^n$ term in $Ph(C_n, q, w)$ vanishes if $q = 2$. If we first take $n \to \infty$ and then set $q = 2$, we have $\lambda_{1,0,j} = \pm\sqrt{w}$, so that the locus $\mathcal{B}_w$ is the union of the semi-infinite real line segments $w > 1$ and $w < -1$. If, on the other hand, we first set $q = 2$, and then vary n, our result (4.32) shows that the limit of $Ph(C_n, 2, w)$ as $n \to \infty$ does not exist, since $Ph(C_n, 2, w)$ is alternatively zero for odd n and $2w^{n/2}$ for even n. If we restrict to odd n, then the problem of the zeros of $Ph(C_n, 2, w)$ is trivial since the function itself vanishes identically, while if we restrict to even n, then the locus $\mathcal{B}_w$ degenerates to a point at $w = 0$, corresponding to the zero at this point with multiplicity $n/2$.

From (2.22) we know that for $q = 0$, $Ph(G, 0, w)$ contains a factor of $(w - 1)$ for an arbitrary graph G, and this is true, in particular, for $G = C_n$. The other zeros occur at real values $w > 1$. For this value $q = 0$, it follows that $\lambda_{1,0,j} = -1 \pm \sqrt{1 - w}$, and these are dominant λ's if $w \neq 1$, so $\mathcal{B}_w$ is the semi-infinite real line segment $w \geq 1$. For $q \neq 0,\ 1,\ 2$ we typically find that the locus $\mathcal{B}_w$ may consist of the union of a (self-conjugate) loop and a line segment. Details depend on the specific value of q.

10 $\mathcal{B}$ for Wheel Graphs

We have also calculated the locus $\mathcal{B}$ for wheel graphs. We denote the $n \to \infty$ limit of the graph Wh_n as $\{Wh\}$. For $w = 1$, $\mathcal{B}_q$ is the unit circle $|q - 2| = 1$, which crosses the real axis at $q = 1$ and $q_c = 3$ and separates the q plane into two regions. In the region with $|q - 2| > 1$, i.e., the region exterior to this circle, $\Phi(\{Wh\}, q, 1) = W(\{Wh\}, q) = q - 2$. In the for which $|q - 2| < 1$, i.e., the region interior to the circle, $|\Phi(\{Wh\}, q, 1)| = |W(\{Wh\}, q)| = 1$.

As w decreases from 1 in the DFCP interval, the boundary $\mathcal{B}_q$ continues to form a closed curve separating the q plane into two regions, as it did for $w = 1$, but there is a discontinuous jump in the crossing point on the left, from $q = 1$ to $q = 2$. This is similar to what we found for the $n \to \infty$ of the circuit graph, where the jump in the crossing point on the left was from $q = 0$ to $q = 1$. The crossing point on the right is

$$q_c = \frac{2(w + 2)}{w + 1} \quad \text{for } \{G\} = \{C\} \tag{10.1}$$

Region R_1 includes the real interval $q > q_c$ and the portion of the complex q plane analytically connected with this interval, and thus lying outside of the boundary $\mathcal{B}_q$. Region R_2 occupies the portion of the q plane inside of the boundary $\mathcal{B}_q$. The dominant λ in region R_1 is $\lambda_{Wh,+}$, while the dominant λ in R_2 is equal to -1. The point q_c occurs where these are degenerate in magnitude. Given this and the relation (3.33), it follows that q_c for $\{G\} = \{Wh\}$ is related to q_c for $\{G\} = \{C\}$, given in (8.5), by replacing q by $q - 1$. That is, if one replaces q_c on the left-hand side of (8.5) by $q_c - 1$ and solves for the new q_c, one obtains (10.1). This is in accord with the fact that the proper q-coloring of the wheel graph with q colors is closely related to the proper coloring of the circuit graph with $q - 1$ colors. As w decreases from 1 to 0 in the DFCP interval the q_c in (10.1) increases continuously from 3 to 4. One can also analyze other ranges of w and the locus $\mathcal{B}_w$ in a similar manner.

11 Some Observations and Conjectures

11.1 Sign Alternation of Successive Terms in $Ph(G,q,w)$

One can write the chromatic polynomial $P(G,q)$ of a graph as

$$P(G,q) = \sum_{j=0}^{n-1} \alpha_{G,n-j}\, q^{n-j}, \qquad (11.1)$$

where, without loss of generality, we take G to be connected. The signs of the coefficients $\alpha_{G,n-j}$ alternate:

$$\mathrm{sgn}(\alpha_{G,n-j}) = (-1)^j, \quad 0 \le j \le n-1. \qquad (11.2)$$

This is proved by iterated application of the deletion-contraction theorem. Since the weighted chromatic polynomial $Ph(G,q,w)$ does not, in general, obey a deletion-contraction theorem, except for the values $w=1$ and $w=0$ for which it reduces to a chromatic polynomial (see (1.1) and (1.2)), one does not expect the coefficients $\alpha_{G,n-j}(w)$ in $Ph(G,q,w)$ to have this sign-alternation property, and they do not. However, from our analysis of weighted coloring polynomials for several families of graphs, we have noticed that for a restricted range of w, namely $0 \le w \le 1$, this sign alternation again holds, namely $\mathrm{sgn}(\alpha_{G,n-j}(w)) = (-1)^j$ for $0 \le j \le n-1$. For $j=n$, namely for the q^0 term in $Ph(G,q,w)$, the sign alternation also holds for $0 \le w < 1$; here the coefficient $\alpha_{G,0}$ contains the factor $(w-1)$ and hence vanishes at $w=1$. We also find that this sign alternation property holds, as far as we have checked it, for real negative w. It is of interest to investigate whether this sign alternation property for $w<1$ holds on other families of graphs. We are currently continuing with this investigation.

Related to this, it is of interest to study where the coefficients $\alpha_{G,n-j}(w)$ vanish in the complex w plane. From our calculation of weighted chromatic polynomials for line and circuit graphs L_n and C_n, we have observed that for the graphs we have considered, the coefficients $\alpha_{L_n,n-j}$ and $\alpha_{C_n,n-j}$ for $1 \le j \le n-1$ have zeros in the real interval $w>1$, while for the coefficients $\alpha_{L_n,0}$ and $\alpha_{C_n,0}$ have, in addition to the always-present zero at $w=1$ (recall (2.22) and (2.33)), the other zeros, if any, again occur in the real interval $w>1$. In contrast, we find that for other graphs, the coefficients $\alpha_{G,n-j}$ may have complex-conjugate pairs of zeros. For example, in $Ph(S_4,q,w)$, the coefficient of the q term, $\alpha_{S_4,1} = w^3-9w^2+27w-20$, has zeros at $w \simeq 1.087$ and $w \simeq 3.9565 \pm 1.6566i$, and the coefficient of the q^0 term, $\alpha_{S_4,0} = -(w-1)(w^2-5w+8)$, has zeros at $w=1$ and $w=(1/2)(5\pm\sqrt{7}i)$. Similarly, coefficients of star graphs S_n for larger n include cases having complex-conjugate pairs of zeros in the w plane.

11.2 Generalized Unimodal Conjecture

From his study of chromatic polynomials, R. Read observed that the magnitudes of the coefficients of successive powers of q^{n-j}, $0 \le j \le n-k(G)$ in a chromatic polynomial satisfy a unimodal property [8]. That is, the magnitudes of these coefficients get successively larger and larger, and then smaller and smaller, as j increases from 0 to $n-k(G)$. There is thus a unique maximal-magnitude coefficient, or two successive coefficients whose magnitudes are equal. From our calculations of weighted chromatic polynomials for a number of families of graphs, we have observed that in the interval $0 \le w \le 1$ this property continues to hold. We therefore state the following conjecture: *Conject.* Let $Ph(G,q,w)$ be written as in

(2.30). Then for real w in the interval $0 \le w \le 1$, the quantities $(-1)^j \alpha_{G,n-j}(w)$, $0 \le j \le n$, are positive and satisfy the unimodal property, i.e., $(-1)^j \alpha_{G,n-j}(w)$ get progressively larger and larger, and a maximal value is reached for a given j, or for two successive j values, and then the quantities $(-1)^j \alpha_{G,n-j}(w)$ get progressively smaller, as j increases from 0 to n.

12 Some Generalizations

Although we have focused in this paper on the proper q-coloring of vertices with one color given a disfavored or favored weighting, we discuss some generalized weighted coloring problems in this section. We first present an extension of (2.4) to the most general case of different fields H_p, $p = 1, \dots, q$, and hence different weighting factors, for each of the q different colors. The generalization to multiple fields corresponding to different spin values in the Potts model was noted, e.g., in [34, 35] and [36] and more recently in [25]. The Hamiltonian for this case is

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} - \sum_{p=1}^{q} \left[H_p \sum_{\ell} \delta_{\sigma_\ell,p} \right]. \tag{12.1}$$

Let us define

$$h_p = \beta H_p, \qquad w_p = e^{h_p} \quad \text{for } 1 \le p \le q \tag{12.2}$$

and denote the set of w_p, $p = 1, \dots, q$ as $\{w\}$. The partition function is a function of q, v, and $\{w\}$, and hence we write it as $Z(G, q, v, \{w\})$. For $v = -1$, the resultant generalized weighted chromatic polynomial is $Ph(G, q, \{w\}) = Z(G, q, -1, \{w\})$.

We have derived the following generalization of the Wu formula for this case (where, as before, G' is a spanning subgraph of G):

$$Z(G, q, v, \{w\}) = \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} \left(\sum_{p=1}^{q} w_p^{n(G'_i)} \right). \tag{12.3}$$

Proof The spins in each component G'_i of G' are connected by edges, so they all have the same value, and there are q possibilities for this value. For a given spanning subgraph G', the weighting factor is the product $\prod_{i=1}^{k(G')} (\sum_{p=1}^{q} w_p^{n(G'_i)})$. This subgraph thus contributes a term $v^{e(G')} \prod_{i=1}^{k(G')} (\sum_{p=1}^{q} w_p^{n(G'_i)})$ to Z. Summing over all spanning subgraphs G' then yields the result (12.3). □

The resultant spanning graph formula for the generalized weighted chromatic polynomial $Ph(G, q, \{w\})$ is obtained by evaluating (12.3) at $v = -1$:

$$Ph(G, q, \{w\}) = \sum_{G' \subseteq G} (-1)^{e(G')} \prod_{i=1}^{k(G')} \left(\sum_{p=1}^{q} w_p^{n(G'_i)} \right). \tag{12.4}$$

Note that some w_p's may disfavor certain color(s), i.e., $0 \le w_p < 1$, while others may favor other color(s), $w_{p'} > 1$. Note also that in the general situation with different H_p, $p = 1, \dots, q$, the dependence of $Ph(G, q, \{w\})$ on q appears via the w_p, $p = 1, \dots, q$ rather than via a polynomial dependence on the variable q.

Let us illustrate this generalization for the case where a set of s colors is subject to a given (disfavored or favored) weighting, i.e. (where without loss of generality, we label these s colors as $1,\ 2,\ldots,s$)

$$H_p = \begin{cases} H \neq 0 & \text{for } 1 \le p \le s \\ 0 & \text{for } s+1 \le p \le q \end{cases} \tag{12.5}$$

so that

$$w_p = \begin{cases} w \neq 1 & \text{for } 1 \le p \le s \\ 1 & \text{for } s+1 \le p \le q. \end{cases} \tag{12.6}$$

Then, with $Z(G,q,v,\{w\})$ written compactly as $Z(G,q,s,v,w)$ in an evident notation, (12.3) takes the form

$$Z(G,q,s,v,w) = \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} \bigl(q - s + s w^{n(G'_i)}\bigr) \tag{12.7}$$

(To avoid awkward notation, we use the same symbol Z for the Potts model partition function with the various sets of arguments, $Z(G,q,v)$, $Z(G,q,v,w)$, $Z(G,q,v,\{w\})$, and $Z(G,q,s,v,w)$.) Writing the weighted chromatic polynomial $Ph(G,q,v,\{w\})$ in the same notation as $Ph(G,q,s,w)$, we have

$$Ph(G,q,s,w) = Z(G,q,s,-1,w). \tag{12.8}$$

With Yan Xu at Stony Brook, a study of the properties of the generalized weighted chromatic polynomial $Ph(G,q,s,w)$ has been carried out, and the results will be reported elsewhere. We note here that since s only appears in (12.7) in the combination

$$\prod_{i=1}^{k(G')} \bigl(q - s + s w^{n(G'_i)}\bigr) = \prod_{i=1}^{k(G')} \left(q + s(w-1) \sum_{r=0}^{n(G'_i)-1} w^r \right), \tag{12.9}$$

it follows that $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ can equivalently be written as polynomials in the variables q, v,

$$t = s(w-1), \tag{12.10}$$

and w. We mention the following general relations involving $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ that hold for $t=0$:

$$Z(G,q,0,v,w) = Z(G,q,s,v,1) = Z(G,q,v), \tag{12.11}$$

$$Ph(G,q,0,w) = Ph(G,q,s,1) = P(G,q), \tag{12.12}$$

and the general relations that hold for $w=0$:

$$Z(G,q,s,v,0) = Z(G,q-s,v), \tag{12.13}$$

and

$$Ph(G,q,s,0) = P(G,q-s). \tag{12.14}$$

From (12.11) and (12.12), it is clear that for $s = 0$, the Z and Ph polynomials reduce to their zero-field forms. A similar reduction to a factor times the zero-field forms occurs if $s = q$:

$$Z(G, q, q, v, w) = w^{n(G)} Z(G, q, v) \tag{12.15}$$

and

$$Ph(G, q, q, w) = w^{n(G)} P(G, q). \tag{12.16}$$

Hence, we are primarily interested in the values $s = 1, \dots, q-1$. Assuming that G contains at least one edge, then, if $q = 1$, it is impossible to satisfy the proper q-coloring constraint, so $Ph(G, 1, s, w) = 0$. This vanishing does not, in general, result as a consequence of an explicit $(q-1)$ factor, unless $s = 0$ or $s = 1$. Instead, when one sets $q = 1$ in $Ph(G, q, s, w)$, one obtains a polynomial with a factor of $s(s-1)(w-1)^2$. Since s is a non-negative integer bounded above by q, the condition that $q = 1$ implies that s is either 0 or 1, and hence this factor must vanish, yielding the necessary result that $Ph(G, 1, s, w) = 0$. One could also study connections with weighted loop models [37, 38].

A different type of generalization is to have the spin-spin couplings depend on the edges of the graph G, so they would be of the form $J_{ij} \equiv J_e$, where i and j denote adjacent vertices of G connected by the edge e. The study of spin models with spin-spin couplings that are different for different lattice directions goes back to the early decades of the twentieth century, reflecting the fact that there are often anisotropies in real magnetic substances. In the 1960's and 1970's, anisotropic spin-spin couplings were studied to investigate how, for various ferromagnetic and antiferromagnetic combinations on different lattices, they could affect critical behavior [39–41]. In the 1970's and later the further generalization to spin-spin couplings that depend on each edge was studied in connection with disordered materials and the question of how such disorder changed the critical behavior [42, 43]; and discussions of edge-dependent J_{ij} continue [25]. The Hamiltonian for the general Potts model for this case, including the full set of q different external fields, is

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \delta_{\sigma_i, \sigma_j} - \sum_{p=1}^{q} \left[H_p \sum_{\ell} \delta_{\sigma_\ell, p} \right]. \tag{12.17}$$

Let us define

$$K_{ij} = \beta J_{ij}, \qquad v_{ij} = e^{K_{ij}} - 1 \tag{12.18}$$

and denote the set of all v_{ij} as $\{v\}$. The partition function is then $Z(G, q, \{v\}, \{w\})$. We give the following general formula for this partition function, where again $G' = (V, E')$ is a spanning subgraph of G:

$$Z(G, q, \{v\}, \{w\}) = \sum_{G' \subseteq G} \left[\prod_{e \in E'} v_e \right] \left[\prod_{i=1}^{k(G')} \left(\sum_{p=1}^{q} w_p^{n(G'_i)} \right) \right]. \tag{12.19}$$

For the weighted proper q-coloring problem, $v_e = -1\ \forall e \in E$, so this generalization reduces to (12.4). Having derived and presented spanning graph formulas for the Potts model partition function and weighted chromatic polynomial for these generalized cases, we focus henceforth on the simple case where only one color is subject to the (disfavored or favored) weighting. Note, as before, that in the general situation, the dependence of $Z(G, q, \{v\}, \{w\})$

on q appears via the w_p, $p = 1, \ldots, q$ rather than via a polynomial dependence on the variable q. We illustrate these generalizations with the circuit graph $G = C_3$. Let us define $\eta_r = \sum_{p=1}^{q} w_p^r$. Then from (12.19) we have

$$Z(C_3, q, \{v\}, \{w\}) = \eta_1^3 + (v_{12} + v_{23} + v_{31})\eta_2\eta_1 + \big[(v_{12}v_{23} + v_{23}v_{31} + v_{31}v_{12}) + v_{12}v_{23}v_{31}\big]\eta_3 \qquad (12.20)$$

and, setting $v_e = -1$ for all of the edges in C_3, we obtain $Ph(C_3, q, \{w\}) = \eta_1^3 - 3\eta_2\eta_1 + 2\eta_3$.

Yet another generalization is to make the sets of colors that one chooses from to assign to each vertex depend on the vertex. With the weighting, this defines a new weighted list-coloring problem. A practical realization of this problem is the allocation of frequencies to radio broadcasting or wireless mobile communication transmitters where each individual transmitter has its own set of available frequencies, no adjacent transmitters should use the same frequency, and there are various disfavored and/or favored frequencies. For a graph $G = (V, E)$, we denote the list of available colors for a given vertex as $\{c_i\}$, where $i = 1, \ldots, n(G)$, and we denote the set of all color lists with the symbol $\{\{c\}\} \equiv \{\{c_1\}, \ldots, \{c_n\}\}$. We define the associated partition function as $Z(G, \{\{c\}\}, \{v\}, \{w\}) = \sum_{\{\sigma_i\}} \exp(-\beta\mathcal{H})$, with

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij}\delta_{\sigma_i,\sigma_j} - \sum_{p=1}^{q}\Big[H_p \sum_{\ell} \delta_{\sigma_\ell,p}\Big], \qquad (12.21)$$

where σ_i takes on values in the list $\{c_i\}$. As before, one may consider special cases of this weighted list coloring problem in which, e.g., the J_{ij} are constants, independent of the edge joining the vertices i and j, and/or where the H_p are of the simple form (12.5), etc. As a simple example, we again take the circuit graph $G = C_3$ and choose the available color lists for each vertex as $\{c_1\} = (1, 2)$, $\{c_2\} = (2, 3)$, $\{c_3\} = (1, 3)$. The generalized weighted chromatic polynomial for this weighted list coloring problem would then be $2w_1w_2w_3$ corresponding to the color assignments $(1, 2, 3)$ and $(2, 3, 1)$ to vertices $i = 1,\ 2,\ 3$. In passing, we recall the case where there are no external fields or corresponding weightings, i.e., $H_p = 0$, so $w_p = 1$ for all $p = 1, \ldots, q$. This is the usual unweighted list coloring problem, as reviewed, e.g., in [44]. For example, for the case $G = C_3$ with the color lists given above, the list chromatic polynomial is 2. In contrast, for $G = C_3$ with color lists $\{c_1\} = (1, 2, 3)$, $\{c_2\} = (1, 2)$, and $\{c_3\} = (1)$ there is only one proper coloring, namely the color assignment $(3, 2, 1)$ to vertices $1,\ 2,\ 3$, so the list coloring polynomial is equal to 1.

13 Conclusions

In this paper we have studied proper q-colorings of the vertices of a graph with a weighting factor w that either disfavors or favors a given color. In particular, we have analyzed a weighted chromatic polynomial $Ph(G, q, w)$ associated with this problem, which generalizes the chromatic polynomial $P(G, q)$. Since $Ph(G, q, w)$ can be obtained as a special limit of the Potts model partition function in an external magnetic field, its study represents a fruitful confluence of statistical mechanics and mathematical graph theory. We have found a number of interesting properties of this weighted chromatic polynomial. Among others, we have shown how it encodes more information about the graph G, as shown by the fact that it is able to distinguish between certain graphs that yield the same chromatic polynomial. We have given formulas for $Ph(G, q, w)$ for various families of graphs G, including line graphs, star graphs, complete graphs, and cyclic lattice strip graphs. For $w \in (0,1)$, $Ph(G,q,w)$ effectively interpolates between $P(G,q)$ and $P(G,q-1)$. Using our results, we have discussed the zeros of $Ph(G,q,w)$ in the q and w planes and their accumulation sets in the limit of infinitely many vertices of G. Finally, we have mentioned some observations, conjectures, and related weighted graph-coloring problems. There is ample motivation for further research on this very interesting subject.

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Appendix A: Tables on Structural Properties

For comparison with our new results for $n_{Ph}(L_y,d)$ and N_{Ph,L_y}, we list here corresponding tables for the following numbers for cyclic strips of the square (sq), triangular (tri), and honeycomb (hc) lattices Λ: (Table 2) $n_P(L_y,d)$ and their sums, $N_{P,L_y,\lambda}$ for the chromatic polynomial with $h=0$, (Table 3) $n_Z(L_y,d)$ and their sums, $N_{Z,L_y,\lambda}$ for the Potts model partition function with $h=0$, and (Table 4) $n_{Zh}(L_y,d)$ and their sums, $N_{Zh,L_y,\lambda}$ for the Potts model partition function with $h \neq 0$ [1, 2].

Table 2 Table of numbers $n_P(L_y,d)$ and their sums, $N_{P,L_y,\lambda}$ for the chromatic polynomial of cyclic strips of the lattice Λ (sq, tri, hc) with $h=0$. Blank entries are zero. See text for further discussion

$L_y \downarrow d \rightarrow$	0	1	2	3	4	5	6	7	8	$N_{P,L_y,\lambda}$
1	1	1								2
2	1	2	1							4
3	2	4	3	1						10
4	4	9	8	4	1					26
5	9	21	21	13	5	1				70
6	21	51	55	39	19	6	1			192
7	51	127	145	113	64	26	7	1		534
8	127	323	385	322	203	97	34	8	1	1500

Table 3 Table of numbers $n_Z(L_y,d)$ and their sums, $N_{Z,G,\lambda}$, for the Potts model partition function on cyclic strips of the lattice Λ (sq, tri, hc) with $h=0$. Blank entries are zero. See text for further discussion

$L_y \downarrow d \rightarrow$	0	1	2	3	4	5	6	7	8	$N_{Z,L_y,\lambda}$
1	1	1								2
2	2	3	1							6
3	5	9	5	1						20
4	14	28	20	7	1					70
5	42	90	75	35	9	1				252
6	132	297	275	154	54	11	1			924
7	429	1001	1001	637	273	77	13	1		3432
8	1430	3432	3640	2548	1260	440	104	15	1	12870

Table 4 Table of numbers $n_{Zh}(L_y, d)$ and their sums, N_{Zh,L_y} for the Potts model partition function on strips of the lattice Λ (sq, tri, hc) with $h \neq 0$. Blank entries are zero. See text for further discussion

$L_y \setminus d$	0	1	2	3	4	5	6	7	8	N_{Zh,L_y}
1	2	1								3
2	5	5	1							11
3	15	21	8	1						45
4	51	86	46	11	1					195
5	188	355	235	80	14	1				873
6	731	1488	1140	489	123	17	1			3989
7	2950	6335	5397	2730	875	175	20	1		18483
8	12235	27352	25256	14462	5530	1420	236	23	1	86515

Appendix B: $Ph(G, q, w)$ for Tree Graphs G

B.1 $n = 5$ Vertices

There are three tree graphs with $n = 5$ vertices, as shown in Fig. 1: (i) the line graph L_5, (ii) the graph Y_5, and (iii) the star graph S_5. The weighted chromatic polynomials for these are

$$\begin{aligned}
Ph(L_5, q, w) &= (q-1)\big[q^4 + (5w-8)q^3 + 3(2w^2 - 9w + 8)q^2 \\
&\quad + (w-2)(w^2 - 16w + 16)q - (w-1)(w^2 - 12w + 16)\big] \\
&= q^5 - (9-5w)q^4 + 2(w-4)(3w-4)q^3 - (-w^3 + 24w^2 - 75w + 56)q^2 \\
&\quad + (-2w^3 + 31w^2 - 76w + 48)q - (1-w)(w^2 - 12w + 16) \qquad \text{(B.1)}
\end{aligned}$$

$$\begin{aligned}
Ph(Y_5, q, w) &= (q-1)(q+w-2)\big[q^3 + 2(2w-3)q^2 + (2w^2 - 13w + 12)q \\
&\quad - (w-1)(3w-8)\big] \\
&= q^5 - (9-5w)q^4 + 2(w-4)(3w-4)q^3 - 2(-w^3 + 13w^2 - 38w + 28)q^2 \\
&\quad + (-5w^3 + 37w^2 - 79w + 48)q - (w-1)(w-2)(8-3w) \qquad \text{(B.2)}
\end{aligned}$$

$$\begin{aligned}
Ph(S_5, q, w) &= (q-1)\big[q^4 + (5w-8)q^3 + 3(2w^2 - 9w + 8)q^2 \\
&\quad + (4w^3 - 24w^2 + 51w - 32)q + (w-1)(w^3 - 7w^2 + 17w - 16)\big] \\
&= q^5 - (9-5w)q^4 + 2(w-4)(3w-4)q^3 \\
&\quad - 2(-2w^3 + 15w^2 - 39w + 28)q^2 \\
&\quad + (w^4 - 12w^3 + 48w^2 - 84w + 48)q \\
&\quad - (w-1)(w^3 - 7w^2 + 17w - 16) \qquad \text{(B.3)}
\end{aligned}$$

B.2 $Ph(G, q, w)$ for Tree Graphs with $n = 6$ Vertices

There are six tree graphs with $n = 6$ vertices, as shown in Fig. 6: (i) the line graph L_6, (ii) the graphY_6, (iii) the graph with a branch in the middle of the line, denoted $iso - Y_6$ (iv) a graph with two branches, denoted H_6, (v) a graph forming a cross, denoted Cr_6, and

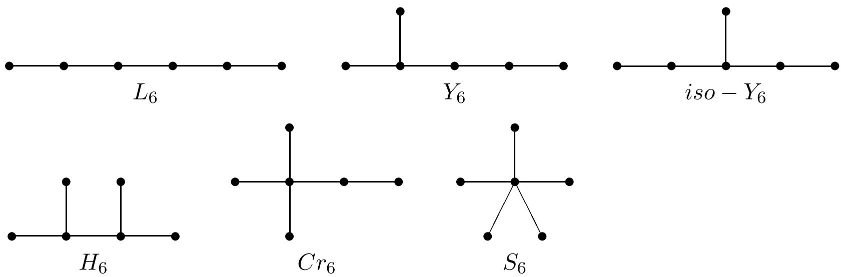

Fig. 6 Tree graphs with $n = 6$

(vi) the star graph S_6. (Again, we order these in terms of graphs with increasing maximal vertex degree Δ; one has $\Delta = 2,\ 3,\ 3,\ 3,\ 4,\ 5$ for graphs (i)–(vi), respectively.) We find

$$\begin{aligned} Ph(L_6,q,w) &= (q-1)\big[q^2+2(w-2)q-3w+4\big]\big[q^3+2(2w-3)q^2 \\ &\quad +(2w^2-13w+12)q-2(w-1)(w-4)\big] \\ &= q^6-(11-6w)q^5+10(w^2-5w+5)q^4 \\ &\quad -2(-2w^3+29w^2-82w+60)q^3 \\ &\quad +(-14w^3+123w^2-264w+160)q^2 \\ &\quad -(-16w^3+113w^2-208w+112)q \\ &\quad +2(1-w)(4-3w)(4-w) \end{aligned} \tag{B.4}$$

$$\begin{aligned} Ph(Y_6,q,w) &= (q-1)\big[q^5+2(3w-5)q^4+2(5w^2-22w+20)q^3 \\ &\quad +(5w^3-50w^2+121w-80)q^2 \\ &\quad +(w^4-16w^3+84w^2-148w+80)q \\ &\quad -(w-1)(w-4)(w^2-7w+8)\big] \\ &= q^6-(11-6w)q^5+10(w^2-5w+5)q^4 \\ &\quad -5(-w^3+12w^2-33w+24)q^3 \\ &\quad +(w^4-21w^3+134w^2-269w+160)q^2 \\ &\quad -(2w^4-28w^3+131w^2-216w+112)q \\ &\quad +(w-1)(w-4)(w^2-7w+8) \end{aligned} \tag{B.5}$$

$$\begin{aligned} Ph(H_6,q,w) &= (q-1)(q-2+w)\big[q^4+(5w-8)q^3+(w-4)(5w-6)q^2 \\ &\quad +(-14w^2+45w-32)q+2(w-1)(5w-8)\big] \\ &= q^6-(11-6w)q^5+10(w^2-5w+5)q^4 \\ &\quad -5(-w^3+12w^2-33w+24)q^3 \\ &\quad +(-19w^3+133w^2-269w+160)q^2 \\ &\quad -(-24w^3+129w^2-216w+112)q \\ &\quad +2(w-1)(w-2)(8-5w) \end{aligned} \tag{B.6}$$

$$\begin{aligned} Ph(H_6,q,w) &= (q-1)(q-2+w)^2\big[q^3+2(2w-3)q^2+(w^2-12w+12)q \\ &\quad -2(w-1)(w-4)\big] \\ &= q^6-(11-6w)q^5+10(w^2-5w+5)q^4 \\ &\quad -2(-3w^3+31w^2-83w+60)q^3 \\ &\quad +(w^4-26w^3+144w^2-274w+160)q^2 \\ &\quad -(w-2)(3w^3-32w^2+84w-56)q \\ &\quad +2(w-1)(w-2)^2(w-4) \end{aligned} \tag{B.7}$$

$$\begin{aligned} Ph(Cr_6,q,w) &= (q-1)(q-2+w)\big[q^4+(5w-8)q^3+(w-4)(5w-6)q^2 \\ &\quad +(2w^3-18w^2+47w-32)q-(w-1)(3w^2-13w+16)\big] \\ &= q^6-(11-6w)q^5+10(w^2-5w+5)q^4 \\ &\quad -(-7w^3+64w^2-167w+120)q^3 \\ &\quad +(2w^4-32w^3+153w^2-278w+160)q^2 \\ &\quad -(5w^4-47w^3+160w^2-229w+112)q \\ &\quad +(w-1)(w-2)(3w^2-13w+16) \end{aligned} \tag{B.8}$$

$$\begin{aligned} Ph(S_6,q,w) &= (q-1)\big[q^5+2(3w-5)q^4+2(5w^2-22w+20)q^3 \\ &\quad +2(5w^3-30w^2+63w-40)q^2 \\ &\quad +(5w^4-40w^3+120w^2-164w+80)q \\ &\quad +(w-1)(w^4-9w^3+31w^2-49w+32)\big] \\ &= q^6-(11-6w)q^5+10(w^2-5w+5)q^4 \\ &\quad -10(-w^3+7w^2-17w+12)q^3 \\ &\quad +5(w^4-10w^3+36w^2-58w+32)q^2 \\ &\quad -(-w^5+15w^4-80w^3+200w^2-245w+112)q \\ &\quad +(1-w)(w^4-9w^3+31w^2-49w+32) \end{aligned} \tag{B.9}$$

Among other things, these calculations can be used to characterize further the way in which the weighted chromatic polynomial is able to distinguish between graphs that yield the same chromatic polynomial. As discussed in the text, all tree graphs with a given number n of vertices yield the same chromatic polynomial, $P(G_{tree,n},q)=q(q-1)^{n-1}$ (and, indeed, also the same Tutte polynomial $T(G_{tree_n},x,y)=x^{n-1}$). Using our results above, we calculate the following differences in weighted chromatic polynomials, relative to $Ph(S_6,q,w)$, for definiteness, from which all other differences can be obtained:

$$\begin{aligned} &Ph(S_6,q,w)-Ph(L_6,q,w) \\ &\quad = w(w-1)^2(q-1)\big[(3q+w)(2q+w)-20q-8w+17\big] \end{aligned} \tag{B.10}$$

$$\begin{aligned} &Ph(S_6,q,w)-Ph(Y_6,q,w) \\ &\quad = w(w-1)^2(q-1)\big(5q^2+4wq+w^2-16q-7w+13\big) \end{aligned} \tag{B.11}$$

$$\begin{aligned} Ph(S_6, q, w) - Ph(H_6, q, w) \\ = w(w-1)^2(q-1)\big[5q^2 + 5wq + w^2 - 16q - 8w + 13\big] \end{aligned} \tag{B.12}$$

$$Ph(S_6, q, w) - Ph(H_6, q, w) = w(w-1)^2(q-1)(2q-3+w)^2 \tag{B.13}$$

$$\begin{aligned} Ph(S_6, q, w) - Ph(Cr_6, q, w) \\ = w(w-1)^2(q-1)\big[3q^2 + 3wq + w^2 - 9q - 5w + 7\big] \end{aligned} \tag{B.14}$$

We thus find that the weighted chromatic polynomials for all of the different n-vertex tree graphs of a given n are, in general, different from each other, although they coincide for $w = 1$ and $w = 0$, where they reduce to chromatic polynomials, and for $q = 1$, where they all vanish.

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